Lecture 3:

Quotient Space

Definition: Let V be a vector space over F and let W be a subspace of V. Let $\vec{v} \in V$. Define:

$$\vec{v} + W = \{ \vec{v} + \vec{w} : \vec{w} \in W \}$$

$\vec{v} + W$ is called a <u>coset of W in V</u>.

Remark: $\vec{v} \in \vec{v} + W$.

Definition: The set V/W (called V mod W), is the set defined by $V/W = \{ \vec{v} + W : \vec{v} \in V \}$

(collection of cosets of W in V)

__Proposition:__ Let $\vec{v}, \vec{v}' \in V$. Then: $\vec{v} + W = \vec{v}' + W$ iff $\vec{v} - \vec{v}' \in W$.

__Proof:__ ($\Rightarrow$) Let $\vec{v} + W = \vec{v}' + W$.

$\because$ $\vec{v} \in \vec{v} + W = \vec{v}' + W$. $\therefore$ $\vec{v} = \vec{v}' + \vec{w}$ for some $\vec{w} \in W$

$\therefore$ $\vec{v} - \vec{v}' = \vec{w} \in W$.

($\Leftarrow$) Suppose $\vec{v} - \vec{v}' \in W$. Suffice to prove that $\vec{v} \in v' + W$.

Let $\vec{w} = \vec{v} - \vec{v}'$. Then: $v = \vec{v}' + w \in \vec{v}' + W$

In fact, $\vec{v} + W \subset \vec{v}' + W$. Similarly, $\vec{v}' = \vec{v} + \vec{w}'$ for some $\vec{w}' \in W$.

$\Rightarrow \vec{v}' + W \subset \vec{v} + W$.

__Definition:__ Define:

$$(\vec{v} + W) + (\vec{v}' + W) \overset{\text{def}}{:=} (\vec{v} + \vec{v}') + W \quad \text{(addition)}$$

$$a \cdot (\vec{v} + W) \overset{\text{def}}{:=} a\vec{v} + W \quad \text{(Scalar multiplication)}$$

Proposition: Suppose $\vec{v} + W = \vec{v}' + W$. Then: for any $\vec{v}'' + W \in V/W$.

- $(\vec{v} + W) + (\vec{v}'' + W) = (\vec{v}' + W) + (\vec{v}'' + W)$
- $a \cdot (\vec{v} + W) = a \cdot (\vec{v}' + W)$ for any $a \in F$.

Proof: Homework!

Remark: Addition and scalar multiplication are well-defined.

Theorem: With addition and scalar multiplication defined above, V/W is a vector space over F, called the quotient space.

Proof: Homework!

Examples of quotient space

$\vec{v}_1 + W = \vec{v}_2 + W$

$\Leftrightarrow \vec{v}_1 - \vec{v}_2 \in W = \{\vec{0}\}$

$\Leftrightarrow \vec{v}_1 = \vec{v}_2$

- Let $W = \{\vec{0}\}$. V/W is the same as V. (isomorphic)

  Let $W = V$. V/V is the same as $\{\vec{0}\}$.

- Let $V = \mathbb{R}^2$. Let W be the y-axis.

  Recall: $(x, y) + W = (x', y') + W$ iff $(x, y) - (x', y') \in W$

  iff $x - x' = 0$

  or $x = x'$

$\therefore$ a vector in V/W is determined by the x-coordinate.

- Let $V = F^\infty$ (infinite sequence)

Let $W \overset{\text{def}}{:=} \{ (0, x_2, x_3, \ldots) : x_i \in F \}$.

As above, two vectors in V/W are the same iff they have the same first coordinate.

$\therefore$ V/W is the same (isomorphic) as F

<u>Remark:</u> Even V and W are infinite dimensional,

V/W is one-dimensional!

Proposition: Suppose V is finite-dimensional. Then:

$$\dim(V/W) = \dim(V) - \dim(W).$$

Proof: Let $\{\vec{w}_1, \dots, \vec{w}_n\}$ be a basis of W.

Extend it to a basis $\{\vec{w}_1, \dots, \vec{w}_n, \vec{v}_1, \dots, \vec{v}_k\}$ of V.

Then: $\dim(W) = n$, $\dim(V) = n+k$

We'll prove that $\{\vec{v}_1 + W, \dots, \vec{v}_k + W\}$ forms a basis of V/W.

If so, we'll have: $\dim(V/W) = k = \underbrace{(n+k)}_{\dim(V)} - \underbrace{k}_{\dim(W)}$

Linear independence:

Suppose: $a_1(\vec{v}_1 + W) + \dots + a_k(\vec{v}_k + W) = W$

$$\Rightarrow (a_1\vec{v}_1 + \dots + a_k\vec{v}_k) + W = W$$

$\therefore\quad a_1\vec{v}_1 + \dots + a_k\vec{v}_k \in W$

$\Rightarrow\ a_1\vec{v}_1 + \dots + a_k\vec{v}_k = b_1\vec{w}_1 + \dots + b_n\vec{w}_n$ for some $b_1, \dots, b_n \in F$.

$\Rightarrow\ a_1\vec{v}_1 + \dots + a_k\vec{v}_k - b_1\vec{w}_1 - \dots - b_n\vec{w}_n = \vec{0}$

As $\{\vec{v}_1, \dots, \vec{v}_k, \vec{w}_1, \dots, \vec{w}_n\}$ is linearly independent,

$a_1 = \dots = a_k = 0$ and $b_1 = \dots = b_n = 0$.

$\therefore\ \{\vec{v}_1 + W, \dots, \vec{v}_k + W\}$ is linear independent.

<u>Span:</u> Let $\vec{v} + W \in V/W$.

Then: $\vec{v} = a_1\vec{w}_1 + \dots + a_n\vec{w}_n + b_1\vec{v}_1 + \dots + b_k\vec{v}_k$ for some a_i's and b_j's.

$$\Rightarrow\ \vec{v} + W = \Big(b_1\vec{v}_1 + \dots + b_k\vec{v}_k + \underbrace{a_1\vec{w}_1 + \dots + a_n\vec{w}_n}_{W}\Big) + W$$

$$= b_1(\vec{v}_1 + W) + \dots + b_k(\vec{v}_k + W)$$

Direct product space

Definition: Let V_1 and V_2 are vector spaces over F. Define:

$$V_1 \times V_2 = \{ (\vec{x}, \vec{y}) : \vec{x} \in V_1 \text{ and } \vec{y} \in V_2 \}$$

(called the direct product of V_1 and V_2)

Define:
- $(\vec{x}_1, \vec{y}_1) + (\vec{x}_2, \vec{y}_2) = (\vec{x}_1 + \vec{x}_2, \vec{y}_1 + \vec{y}_2)$ for $\forall \vec{x}_1, \vec{x}_2 \in V_1$, $\vec{y}_1, \vec{y}_2 \in V_2$
- $a(\vec{x}, \vec{y}) = (a\vec{x}, a\vec{y})$ for $\forall a \in F$, $\vec{x} \in V_1$, $\vec{y} \in V_2$.

Then: $V_1 \times V_2$ forms a vector space over F.

Theorem: $\dim(V_1 \times V_2) = \dim(V_1) + \dim(V_2)$

Proof: (Idea) Let $\beta_1 = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ = basis for V_1.
$\beta_2 = \{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_m\}$ = basis for V_2.

Then: $\{(\vec{v}_1, \vec{0}_2), \ldots, (\vec{v}_n, \vec{0}_2), (\vec{0}_1, \vec{w}_1), \ldots, (\vec{0}_1, \vec{w}_m)\}$ forms a basis for $V_1 \times V_2$ (where $\vec{0}_1$ is the zero in V_1
$\vec{0}_2$ is the zero in V_2]

[Check! Exercise)

Existence of basis

For a finite-dimensional vector space, the basis can be constructed as follows:

$\neq \vec{0}$

$\{\vec{v}_1\}$ $\leftarrow$ Linear independent

$\downarrow$

$\{\vec{v}_1, \vec{v}_2\}$ $\leftarrow$ Attach one more vector $\vec{v}_2$ $\ni$ $\{\vec{v}_1, \vec{v}_2\}$ is linearly independent

$\downarrow$

$\vdots$

$\{\vec{v}_1, \dots, \vec{v}_n\}$ $\leftarrow$ must stop at some point.

Constructive proof for the existence of basis.

Zorn's lemma Assume $\mathcal{C}$ is a collection of subsets of some fixed unnamed set. Assume that for all chain:

$$S_1 \subset S_2 \subset \cdots \quad \text{in } \mathcal{C}, \text{ we have } \bigcup_i S_i \in \mathcal{C}$$

$$\underset{\mathcal{C}}{\overset{\cap}{S_1}} \qquad \underset{\mathcal{C}}{\overset{\cap}{S_2}}$$

Then: $\mathcal{C}$ contains a maximal element.

(That is, $\exists\, M \in \mathcal{C} \ni$ there is no set $\tilde{M} \in \mathcal{C}$ satisfying: $M \subset \tilde{M}$

$\uparrow$ properly contained)