Lecture 23

Recall:

Given $T: V \to V$, V is fin-dim.

Find a basis β of V $\ni$ $[T]_\beta = \begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_k \end{pmatrix}$

(A_i = block square matrix)

$A_i = \begin{pmatrix} \lambda & 1 & & 0 \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix}$ where λ is an eigenvalue of T.

<u>Remark</u>:
- $[T]_\beta$ is called the <u>Jordan Canonical form</u> of T
- A_i is called a <u>Jordan block</u> corresponding to λ
- β is called the <u>Jordan canonical basis</u>.

- Cycle : $\{(T-\lambda I)^{p-1}\vec{x},\ (T-\lambda I)^{p-2}\vec{x},\ \ldots\ ,\ \vec{x}\}$

initial vector of a cycle

end vector of a cycle

Goal: Find a basis of V = disjoint union of cycles.

Properties of K_λ (Proof: later!)

① K_λ is T-invariant.

② $E_\lambda \subseteq K_\lambda$

③ Let $T: V \to V$ (V = fin-dim) and char poly splits. Then:

- $V = K_{\lambda_1} \oplus K_{\lambda_2} \oplus \dots \oplus K_{\lambda_k}$
- $\dim(K_{\lambda_i}) = m_i$ = multiplicity of λ_i

④ If $\beta = \gamma_1 \cup \gamma_2 \cup \dots \cup \gamma_k$ (γ_i = cycles) is a basis of V, then β is a Jordan Canonical basis.

⑤ Each K_λ has a basis consisting of unions of cycles!

⑤ Let $\gamma_1 = \{ (T-\lambda_1 I)^{p_1-1}(\vec{v}_1), \ldots, \vec{v}_1 \}$
$\vdots \qquad\qquad \vdots$
$\gamma_q = \{ (T-\lambda_q I)^{p_q-1}(\vec{v}_q), \ldots, \vec{v}_q \}$
cycles.

If initial vectors are linearly independent, then: $\gamma_1, \gamma_2, \ldots, \gamma_q$ are disjoint and $\gamma_1 \cup \gamma_2 \cup \ldots \cup \gamma_q$ is linearly independent.

⑥ Put $q = 1$. We get: every cycle is linearly independent

⑦ (Main Thm). Let $T: V \to V$ (fin-dim) and char poly splits. Then, $\exists$ a Jordan Canonical basis for V.

Example: Let $A = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 3 \end{pmatrix}$

Char poly of $A = (2-t)^4$. $\therefore$ $\lambda = 2$ is the ONLY eigenvalue.

$\therefore$ $\dim(K_\lambda) = 4$. $\leftarrow$ multiplicity.

Need to find : disjoint union of cycles

4 possibilities: ① One cycle of length 4 ✓

② two cycles of length $\begin{cases} 1+3 & \times \\ 2+2 & \times \end{cases}$

③ Three cycles of length $1+1+2$ ×

④ Four cycles of length $1+1+1+1$ ×

We can check $\dim(E_\lambda) = 1$

Go ahead to find a cycle $= \{ (A-\lambda I)^3 \vec{v}, \ldots, \vec{v} \}$

Theoretical proof: $T: V \to V$ (fin-dim), char poly splits.

Claim 1: $(T - \lambda_i I)\big|_{K_{\lambda_j}} : K_{\lambda_j} \to K_{\lambda_j}$ is 1-1 if $i \neq j$.

Pf: Let $\vec{x} \in K_{\lambda_j}$, $\vec{x} \neq \vec{0}$, and $(T - \lambda_i I)(\vec{x}) = \vec{0}$.

Let p = smallest integer such that $(T - \lambda_j I)^p (\vec{x}) = \vec{0}$.

Let $\vec{y} = (T - \lambda_j I)^{p-1}(\vec{x}) \neq \vec{0}$. Then: $(T - \lambda_j I)(\vec{y}) = (T - \lambda_j I)^p(\vec{x}) = \vec{0}$.

$\therefore \vec{y} \in E_{\lambda_j}$.

Now, $(T - \lambda_i I)(\vec{y}) = (T - \lambda_i I)(T - \lambda_j I)^{p-1}(\vec{x})$

$= (T - \lambda_j I)^{p-1} \cancel{(T - \lambda_i I)(\vec{x})} = \vec{0}$.

$\therefore \vec{y} \in E_{\lambda_i}$

$\therefore \vec{y} \in E_{\lambda_i} \cap E_{\lambda_j} = \{\vec{0}\}$ $\quad \therefore \vec{y} = \vec{0}$. Contradiction.

$\therefore N\left((T - \lambda_i I)\big|_{K_{\lambda_j}}\right) = \{\vec{0}\}$ $\therefore$ 1-1

<u>Remark:</u> K_{λ_j} is T-invariant and $(T - \lambda_i I)$-invariant.

<u>Claim 2:</u> $\dim(K_{\lambda_i}) \leq m_i$ = multiplicity of λ_i and $K_{\lambda_i} = N((T - \lambda_i I)^{m_i})$.

<u>Pf:</u> ① Let $g(t)$ = char poly of $T|_{K_{\lambda_i}}$.

Then: $g(t) \mid$ char poly of T.

Now, $(T - \lambda_j I)|_{K_{\lambda_i}}(\vec{x}) \neq \vec{0}$ if $\lambda_j \neq \lambda_i$ and $\vec{x} \neq \vec{0}$.

$\therefore$ λ_i is the ONLY eigenvalue of $T|_{K_{\lambda_i}}$.

$\therefore$ $g(t) = (\lambda_i - t)^d$, $d = \dim(K_{\lambda_i})$

$\therefore$ $d \leq m_i$

② $N\left((T-\lambda_i I)^{m_i}\right) \subseteq K_{\lambda_i}$ (by definition)

Now, $g\left(T|_{K_{\lambda_i}}\right) = 0$ by Caley-Hamiton Thm,

char poly of $T|_{K_{\lambda_i}}$

$\therefore \left(T|_{K_{\lambda_i}} - \lambda_i I\right)^d = 0 \Rightarrow (T-\lambda_i I)^d(\vec{x}) = \vec{0}$ for $\forall \vec{x} \in K_{\lambda_i}$

$\Rightarrow (T-\lambda_i I)^{m_i}(\vec{x}) = \vec{0}$ for $\forall \vec{x} \in K_{\lambda_i}$

$\therefore K_{\lambda_i} \subseteq N\left((T-\lambda_i I)^{m_i}\right)$.

Claim 3: $V = K_{\lambda_1} + K_{\lambda_2} + \dots + K_{\lambda_k}$

Pf: By M.I. on k = # of distinct eigenvalues

When $k=1$, let m = multiplicity of λ_1. Then, char poly of T " $(\lambda_1 - t)^m$

By Caley-Hamilton Thm, $g(T) = (\lambda_1 I - T)^m = 0 \leftarrow$ zero transf.

$\therefore \quad K_{\lambda_1} = N((T - \lambda_1 I)^m) = V$

$\therefore$ Thm is true for $k=1$.

Assume the thm is true for transf. w/ fewer than k eigenvalues.

We'll show the thm is true for k distinct eigenvalues.

Claim 4: Let $W = R\left((T-\lambda_k I)^m\right)$ ← multiplicity of λ_k

Then: (1) $T|_W : W \to W$ is well-defined.

(2) $T|_W$ has $k-1$ distinct eigenvalues: $\lambda_1, \lambda_2, \dots, \lambda_{k-1}$

(3) $(T-\lambda_k I)^m\big|_{K_{\lambda_i}} : K_{\lambda_i} \to K_{\lambda_i}$ is onto $(i<k)$

Assume Claim 4 is true. Let $\vec{x} \in V$. Then: $(T-\lambda_k I)^m \vec{x} \in W$

By induction hypothesis, $\exists\ \vec{w}_i \in K'_{\lambda_i}$ = generalized eigenspace of λ_i of $T|_W$.

Such that $(T-\lambda_k I)^m \vec{x} = \vec{w}_1 + \vec{w}_2 + \dots + \vec{w}_{k-1}$

Easy to check: $K'_{\lambda_i} \subseteq K_{\lambda_i}$ for $i<k$.

Since $(T-\lambda_k I)^m\big|_{K_{\lambda_i}} : K_{\lambda_i} \to K_{\lambda_i}$ is onto, then:

for each $\vec{w}_i \in K'_{\lambda_i}$, $\exists\, \vec{v}_i \in V \ni (T-\lambda_k I)^m(\vec{v}_i) = \vec{w}_i$

$$\therefore\ (T-\lambda_k I)^m(\vec{x}) = (T-\lambda_k I)^m(\vec{v}_1) + \dots + (T-\lambda_k I)^m(\vec{v}_{k-1})$$

$$\iff (T-\lambda_k I)^m(\vec{x} - \vec{v}_1 - \vec{v}_2 - \dots - \vec{v}_{k-1}) = \vec{0}$$

$$\therefore\ \overbrace{\vec{x} - \vec{v}_1 - \vec{v}_2 - \dots - \vec{v}_{k-1}}^{=\,\vec{v}_k} \in N\left((T-\lambda_k I)^m\right) = K_{\lambda_k}$$

$$\therefore\ \vec{x} = \underset{\substack{\in \\ K'_{\lambda_1} \\ \subseteq \\ K_{\lambda_1}}}{\vec{v}_1} + \underset{\substack{\in \\ K_{\lambda_2}}}{\vec{v}_2} + \dots + \underset{\substack{\in \\ K_{\lambda_{k-1}}}}{\vec{v}_{k-1}} + \underset{\substack{\in \\ K_{\lambda_k}}}{\vec{v}_k}$$

By M.I, the thm is true.

Proof of Claim 4

Let $W = R((T-\lambda_k I)^m)$. $\because$ T and $(T-\lambda_k I)^m$ commutes

$\therefore$ W is T-invariant and $T|_W$ is well-defined.

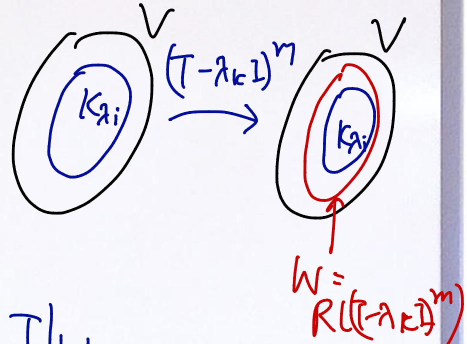
Consider $(T-\lambda_k I)^m|_{K_{\lambda_i}} : K_{\lambda_i} \to K_{\lambda_i}$

We'll prove : $(T-\lambda_k I)^m|_{K_{\lambda_i}}$ is onto. $(i < k)$

Note: $(T-\lambda_k I)|_{K_{\lambda_i}}$ is 1-1 and so onto.

$\therefore$ $(T-\lambda_k I)^m|_{K_{\lambda_i}}$ is also onto.

Also, $E_{\lambda_i} \subseteq K_{\lambda_i} \subseteq W = R((T-\lambda_k I)^m)$



$\therefore$ λ_i is an eigenvalue of $T|_W$.

$\therefore$ $\lambda_1, \lambda_2, \ldots, \lambda_{k-1}$ are eigenvalues of $T|_W$.

Next, suppose λ_k is an eigenvalue of $T|_W$

Suppose $T|_W(\vec{v}) = \lambda_k \vec{v}$ for $\vec{v} \neq \vec{0} \in W = R\left((T-\lambda_k I)^m\right)$

Then: $\vec{v} = (T-\lambda_k I)^m(\vec{y})$, $\vec{y} \in V$,

Thus, $\vec{0} = (T-\lambda_k I)\vec{v} = (T-\lambda_k I)^{m+1}(\vec{y})$

$\Rightarrow \vec{y} \in K_{\lambda_k} = N\left((T-\lambda_k I)^m\right)$

$$\therefore \underbrace{(T-\lambda_k I)^m(\vec{y})}_{\vec{v}} = \vec{0}$$

Contradiction ($\because \vec{v} \neq \vec{0}$)

$\therefore$ $T|_W$ has ONLY $k-1$ distinct eigenvalues: $\lambda_1, \lambda_2, \ldots, \lambda_{k-1}$.

<u>Claim 5</u>: Let β_i = ordered basis of K_{λ_i}.

Then: $\beta = \beta_1 \cup \beta_2 \cup \dots \cup \beta_k$ is a disjoint union and a basis of V.

<u>Pf</u>: <u>Disjoint union</u>: Let $\vec{x} \in \beta_i \cap \beta_j$ $(i \neq j) \subseteq K_{\lambda_i} \cap K_{\lambda_j}$

$$K_{\lambda_j} \ni (T - \lambda_i I)(\overset{\neq \vec{0}}{\underset{K_{\lambda_j}}{\vec{x}}}) \neq \vec{0} \qquad (\because\ T - \lambda_i I\big|_{K_{\lambda_j}} \text{ is 1-1})$$

$$(T - \lambda_i I)^2(\vec{x}) \neq \vec{0}$$

$$\vdots$$

$$(T - \lambda_i I)^p(\vec{x}) \neq 0 \quad \text{for } \forall p.$$

$\therefore\ \vec{x} \notin K_{\lambda_i}$ (Contradiction) $\qquad \therefore\ \beta_i \cap \beta_j = \phi$.

<u>Basis</u> Let $\vec{x} \in V$. By Claim 3,

$$\vec{x} = \vec{v}_1 + \vec{v}_2 + \dots + \vec{v}_k \,, \quad \text{where } \vec{v}_i \in K_{\lambda_i}.$$

$\therefore$ $\vec{x}$ is a lin. combination of vectors in β

$\therefore$ $V = \operatorname{span}(\beta)$

alg. mult. of λ_i

Let $q = |\beta|$. Then: $\dim(V) \le q$

Let $d_i = \dim(K_i)$. Then: $q = \sum_i d_i \le \sum_i m_i = \dim(V)$

$\therefore$ $q = \dim(V) \Rightarrow \beta$ is a basis.

Claim 6: $\dim(K_{\lambda_i}) = m_i$

Pf:

$$\sum_i d_i = \sum m_i \Rightarrow \sum_i \big(\overbrace{m_i - d_i}^{\geq 0}\big) = 0$$

$$\dim(V) \Rightarrow m_i = d_i \text{ for } \forall i.$$

Claim 7: Let $\gamma_1 = \{ (T - \lambda I)^{m_1} \vec{v}_1, \ldots, \vec{v}_1 \}$

$\vdots$

$\gamma_q = \{ (T - \lambda I)^{m_q} \vec{v}_q, \ldots, \vec{v}_q \}$

If initial vectors are linearly independent, then:

$\gamma = \gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_q$ is disjoint union and it's linearly independent.

Pf: Disjoint union: Exercise.

Linear independence: Use M.I. on $n = \#$ of elements in γ.

When $n=1$, trivial.

Assume the thm is true when γ has less than n elements.

When $|\gamma| = n$, let

$$\gamma_1' = \{ (T-\lambda I)^{m_1}\vec{v}_1, \; \dots, \; (T-\lambda I)\vec{v}_1 \}$$

$$\vdots$$

$$\gamma_q' = \{ (T-\lambda I)^{m_q}\vec{v}_q, \; \dots, \; (T-\lambda I)\vec{v}_q \}$$

Let $\gamma' = \gamma_1' \cup \gamma_2' \cup \dots \cup \gamma_q'$ $\quad |\gamma'| = n-q$

Let $W = \text{span}(\gamma)$. Let $U = (T-\lambda I)|_W$

Then: $\quad R(U) = \text{span}(\gamma')$ (Check)

$\because$ initial vectors of γ_i's are L.I.

$\therefore$ γ' is L.I. (by induction hypothesis)

Thus, $\dim(R(U)) = n-q$.

Also, $S = \{ (T-\lambda I)^{m_1}\vec{v}_1, \ldots, (T-\lambda I)^{m_q}(\vec{v}_q) \} \subseteq N(U)$

$\therefore \dim(N(U)) \geq q$

$\therefore n \geq \dim(W) = \dim(R(U)) + \dim(N(U))$

$\geq (n-q) + q = n$

$\therefore \dim(W) = n$, $|\gamma| = n$ and $\text{span}(\gamma) = W$

$\therefore \gamma$ is a basis and γ is L.I.

<u>Claim 8:</u> Suppose β = basis of V = disjoint union of cycles.

Then: ① For each cycle γ in β, $W = \text{span}(\gamma]$ is T-invariant and $[T|_W]_\gamma$ = Jordan block.

② β = JC basis for V.

<u>Pf:</u> Done before (not in the dream)

Claim 9: Let λ = eigenvalue of T.

Then: K_λ has a basis β = union of disjoint cycles w.r.t. λ.

Pf: By M.I. on $n = \dim(K_\lambda)$

When $n=1$, trivial.

Suppose the result is true for $\dim(K_\lambda) < n$.

When $\dim(K_\lambda) = n$. Let $U = (T - \lambda I)|_{K_\lambda}$

Then: $\dim(R(U)) < \dim(K_\lambda) = n$

$$\left(\because \ \dim(K_\lambda) = \dim(\underset{\substack{\cup\!\shortmid \\ E_\lambda}}{N(U)}) + \dim(R(U)) \quad (\dim(E_\lambda) \geq 1) \right)$$

Let K_λ' = generalized eigenspace corresponding to λ of $T|_{R(U)}$

Easy to check $R(U) = K_\lambda'$ (Check)

By induction hypothesis, $\exists$ disjoint cycles $\gamma_1, \gamma_2, \ldots, \gamma_q$ of $T|_{R(U)} \ni \gamma = \bigcup_{i=1}^{q} \gamma_i$ is a basis for $R(U) = K_{\lambda}'$.

Let $\gamma_i = \{ (T|_{R(U)} - \lambda I)^{m_i} x_i, \ldots, \underset{\substack{\wedge \\ R(U)}}{x_i} \}$

Now, let $x_i = U\vec{v}_i = (T - \lambda I)\vec{v}_i$, $\vec{v}_i \in K_\lambda$

Define: $\tilde{\gamma}_i = \{ \underbrace{(T-\lambda I)^{m_i+1}(\vec{v}_i)}_{= \vec{w}_i}, \ldots, (T-\lambda I)(\vec{v}_i), \vec{v}_i \}$

Note: $\bigcup_{i=1}^{q} \gamma_i$ is L.I., so,

$S = \{ \vec{w}_1, \vec{w}_2, \ldots, \vec{w}_q \}$ is L.I. subset of E_λ.

Extend S to a basis of E_λ: $\{ \vec{w}_1, \vec{w}_2, \ldots, \vec{w}_q, \vec{u}_1, \ldots, \vec{u}_s \}$

By construction, $\tilde{\gamma}_1, \tilde{\gamma}_2, \ldots, \tilde{\gamma}_q, \{\vec{u}_1\}, \{\vec{u}_2\}, \ldots, \{\vec{u}_s\}$ are disjoint union of cycles $\ni$ initial vectors are lin. independent.

$\therefore \quad \tilde{\gamma} = \bigcup_{i=1}^{q} \tilde{\gamma}_i \cup \{\vec{u}_1\} \cup \dots \cup \{\vec{u}_s\}$ is L.I. subset of K_λ.

Now, we show that $\tilde{\gamma}$ is a basis of K_λ.

Suppose $|\gamma| = r = \dim(R(U))$

basis of $R(U)$

Then: $|\tilde{\gamma}| = r + q + s$

$\therefore \{\vec{w}_1, \vec{w}_2, \dots, \vec{w}_q, \vec{u}_1, \dots, \vec{u}_s\}$ is a basis for $E_\lambda = N(U) = N((T - \lambda I)|_{K_\lambda})$

$\therefore \quad \dim(N(U)) = q + s$

$$\therefore \quad \dim(K_\lambda) = \underbrace{\dim(R(U))}_{= r} + \underbrace{\dim(N(U))}_{= q+s} = |\tilde{\gamma}|$$

$\therefore \quad \tilde{\gamma}$ is a basis for K_λ.

<u>Claim 10:</u> T has JCF.