Lecture 2: **Direct Sum:** Let U and W be subspaces of V.

Then: $U + W = \{\vec{x} + \vec{y} : \vec{x} \in U \text{ and } \vec{y} \in W\}$ is also a subspace of V (Check!)

Definition: V is said to be the **direct sum** of U and W, denoted by $V = U \oplus W$ if $V = U + W$ and $U \cap V = \{\vec{0}\}$.

Lemma: $V = U \oplus W$ iff for $\forall \vec{v} \in V$, $\exists !$ vectors $\vec{u} \in U$ and $\vec{w} \in W \ni \vec{v} = \vec{u} + \vec{w}$

Proof: $(\Rightarrow)$ If $\vec{v} \in V$, then $\vec{v} = \vec{u} + \vec{w}$ for some $\vec{u} \in U$ and $\vec{w} \in W$ ($\because V = U \oplus W$)

For uniqueness, let $\vec{v} = \vec{u}_1 + \vec{w}_1 = \vec{u}_2 + \vec{w}_2$.

Then: $\vec{u}_1 - \vec{u}_2 = \vec{w}_2 - \vec{w}_1 \in U \cap W = \{\vec{0}\}$. $\therefore$ $\vec{u}_1 - \vec{u}_2 = \vec{0} \Rightarrow \vec{u}_1 = \vec{u}_2$

$\vec{w}_2 - \vec{w}_1 = \vec{0} \Rightarrow \vec{w}_1 = \vec{w}_2$

$(\Leftarrow)$ $V = U + W$ is obvious.

Now, let $\vec{z} \in U \cap W$. $\exists!\ \vec{u}$ and $\vec{w} \ni \vec{z} = \vec{u} + \vec{w}$.

Then: $\vec{z} = \underset{\substack{\rotatebox{90}{$\in$} \\ U}}{\vec{z}} + \underset{\substack{\rotatebox{90}{$\in$} \\ W}}{\vec{0}} = \underset{\substack{\rotatebox{90}{$\in$} \\ U}}{\vec{0}} + \underset{\substack{\rotatebox{90}{$\in$} \\ W}}{\vec{z}}$ $\Rightarrow$ $\vec{u} = \vec{0}$ and $\vec{w} = \vec{0}$

$\Rightarrow$ $\vec{z} = \vec{0}$.

$\therefore$ $U \cap W = \{\vec{0}\}$

Projection operators

Definition: Suppose $V = U \oplus W$. Define: $P : V \to U$ as follows:

For any $\vec{v} \in V$, write $\vec{v} = \vec{u} + \vec{w}$ where $\vec{u} \in U$ and $\vec{w} \in W$.

Then: define: $P(\vec{v}) = \vec{u}$

Remark:

1. P is well-defined
2. $P \circ P = P$

Definition V is said to be a direct sum of subspaces $U_1, U_2, \dots, U_k$, denoted as $V = U_1 \oplus U_2 \oplus \dots \oplus U_k$, if for $\forall \vec{v} \in V$, $\exists!$ vectors $\vec{u}_i \in U_i$ $(1 \le i \le k)$ $\ni$ $\vec{v} = \vec{u}_1 + \vec{u}_2 + \dots + \vec{u}_k$.

Remark:

- $U_1 \oplus \dots \oplus U_k = (\cdots((U_1 \oplus U_2) \oplus U_3) \oplus \cdots \oplus U_k)$
- $V = U_1 \oplus U_2 \oplus \dots \oplus U_k$ iff :

  1. $V = U_1 + U_2 + \dots + U_k$
  2. $U_r \cap \sum_{i \ne r} U_i = \{\vec{0}\}$ for $1 \le r \le k$.

Dimension of direct sum

Theorem: Let V be a finite-dim vector space. $U_1, U_2, \ldots, U_m$ are subspaces of V. Then:

$$\dim(U_1 \oplus U_2 \oplus \ldots \oplus U_m) = \sum_{i=1}^{m} \dim(U_i)$$

Proof: Let β_i = basis of U_i for $i = 1, 2, \ldots, m$.

Let $\beta = \beta_1 \,\dot\cup\, \beta_2 \,\dot\cup \ldots \dot\cup\, \beta_m$ (disjoint union)

For $\forall \vec{v} \in U_1 \oplus \ldots \oplus U_m$, $\exists!\ \vec{u}_1 \in U_1, \vec{u}_2 \in U_2, \ldots, \vec{u}_m \in U_m \Rightarrow$

$$\vec{v} = \vec{u}_1 + \ldots + \vec{u}_m.$$

Each $\vec{u}_i$ can be written as a linear combination of elements in β_i.

$\therefore\ \mathrm{Span}(\beta) = U_1 \oplus \ldots \oplus U_m$

β is linear independent.

Let $\vec{0} = (a_1^1 u_1^1 + a_2^1 u_2^1 + \ldots + a_n^1 u_n^1) + \ldots + (a_1^m u_1^m + \ldots + a_n^m u_n^m)$

$u_1^1 \in \beta_1$, $u_2^1 \in \beta_1$, $u_n^1 \in \beta_1$, $u_1^m \in \beta_m$, $u_n^m \in \beta_m$

Then: each $a_1^j u_1^j + \ldots + a_n^j u_n^j = \vec{0}$ for $\forall j$

$\Rightarrow a_1^j = a_2^j = \ldots = a_n^j = 0$ for $\forall j$.

$\therefore$ β is linear independent.

$\therefore$ β is a basis.

$\therefore \dim(U_1 \oplus \ldots \oplus U_m) = |\beta| = \sum_{i=1}^{m} \dim(U_i)$.

Remark: In general,

$$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$$

(Homework!)