Lecture 18:

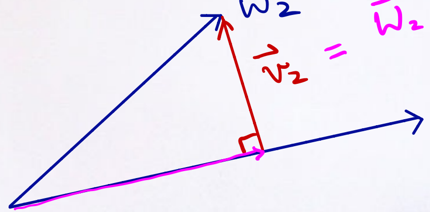
Prop: Let V be an inner product space and $S_n = \{\vec{w}_1, \ldots, \vec{w}_n\}$ be a linearly independent subset of V. Define:

$$S_n' = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\} \quad \text{where} \quad \vec{v}_1 = \vec{w}_1 \quad \text{and}$$

for $k = 2, \ldots, n$,

$$\vec{v}_k \overset{\text{def}}{=} \vec{w}_k - \sum_{j=1}^{k-1} \left(\frac{\langle \vec{w}_k, \vec{v}_j \rangle}{\|\vec{v}_j\|^2} \right) \vec{v}_j$$

Then: S_n' is orthogonal and $\operatorname{Span}(S_n') = \operatorname{span}(S_n)$



$\vec{w}_2 - c_1 \vec{v}_1$ where $c_1 = \dfrac{\langle \vec{w}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}$

Proof: We prove by induction on n.

For $n=1$, we simply have $S_1' = S_1$. $\therefore$ The statement is obviously true.

Suppose the statement is true for $n = m-1$.

That's, $S'_{m-1} = \{\vec{v}_1, \dots, \vec{v}_{m-1}\}$ is orthogonal and

$$\left.\begin{array}{l} S'_{m-1} = \{\vec{v}_1, \dots, \vec{v}_{m-1}\} \text{ is orthogonal and} \\ \operatorname{Span}(S'_{m-1}) = \operatorname{Span}(S_{m-1}) \end{array}\right\} \text{Induction hypothesis}$$

Now, consider a lin. independent subset $S_m = \{\vec{w}_1, \vec{w}_2, \dots, \vec{w}_{m-1}, \vec{w}_m\}$

Then: for $\vec{v}_m \stackrel{\text{def}}{=} \vec{w}_m - \sum_{j=1}^{m-1} \frac{\langle \vec{w}_m, \vec{v}_j \rangle}{\|\vec{v}_j\|^2} \vec{v}_j$, we have:

$$\langle \vec{v}_m, \vec{v}_i \rangle = \langle \vec{w}_m, \vec{v}_i \rangle - \sum_{j=1}^{m-1} \frac{\langle \vec{w}_m, \vec{v}_j \rangle}{\|\vec{v}_j\|^2} \langle \vec{v}_j, \vec{v}_i \rangle \quad \text{for } i = 1, \dots, m-1$$

$$= \cancel{\langle \vec{w}_m, \vec{v}_i \rangle} - \cancel{\frac{\langle \vec{w}_m, \vec{v}_i \rangle}{\|\vec{v}_i\|^2} \langle \vec{v}_i, \vec{v}_i \rangle} = 0$$

$\therefore$ S_m' is orthogonal.

Also, $\vec{v}_m \neq \vec{0}$ since otherwise, $\vec{w}_m \in \operatorname{Span}(S_{m-1}')$

$$= \operatorname{Span}(S_{m-1}) = \operatorname{Span}(\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_{m-1}\})$$

contradicting the condition that S_m is linearly independent.

Hence, $S_m' = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{m-1}, \vec{v}_m\}$ is orthogonal subset consisting of non-zero vectors. $\therefore$ S_m' is linearly independent.

Also, $\underbrace{\operatorname{Span}(S_m')}_{\dim = m} \subset \underbrace{\operatorname{Span}(S_m)}_{\dim = m} \Rightarrow \operatorname{Span}(S_m') = \operatorname{Span}(S_m)$

The above construction of an orthogonal basis is called <u>Gram-Schmidt process.</u>

Orthogonal complement

Def: Let S be a non-empty subset of an inner product space V. The orthogonal complement of S is defined as:

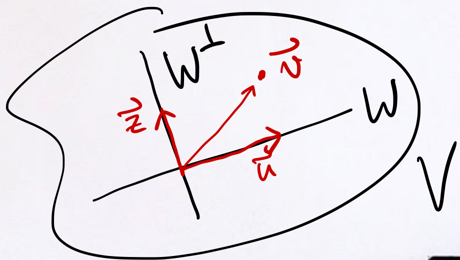
$$S^{\perp} \overset{\text{def}}{=} \{ \vec{x} \in V : \langle \vec{x}, \vec{y} \rangle = 0 \text{ for } \forall \vec{y} \in S \}$$

Proposition: Let V be an inner product space and $W \subset V$ a finite-dim subspace of V. Then: $\forall \vec{y} \in V$, $\exists ! \, \vec{u} \in W$ and $\vec{z} \in W^{\perp}$ such that $\vec{y} = \vec{u} + \vec{z}$.

Furthermore, if $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ is an orthonormal basis for W, then:

$$\vec{u} = \sum_{i=1}^{k} \langle \vec{y}, \vec{v}_i \rangle \vec{v}_i$$

The vector $\vec{u} \in W$ is called the orthogonal projection of $\vec{y}$ on W.



Proof: Given $\vec{y} \in V$, we set $\vec{u} \overset{\text{def}}{=} \sum_{i=1}^{k} \langle \vec{y}, \vec{v}_i \rangle \vec{v}_i \in W$

and $\vec{z} \overset{\text{def}}{=} \vec{y} - \vec{u}$. Then: $\vec{y} = \underset{\substack{\uparrow \\ W}}{\vec{u}} + \underset{\substack{\uparrow \\ W^{\perp}\,??}}{\vec{z}}$.

Now, $\langle \vec{z}, \vec{v}_j \rangle = \langle \vec{y} - \vec{u}, \vec{v}_j \rangle = \langle \vec{y}, \vec{v}_j \rangle - \langle \vec{u}, \vec{v}_j \rangle$

$$= \langle \vec{y}, \vec{v}_j \rangle - \underbrace{\sum_{i=1}^{k} \langle \vec{y}, \vec{v}_i \rangle \langle \vec{v}_i, \vec{v}_j \rangle}_{\langle \vec{y}, \vec{v}_j \rangle}$$

$$\therefore \Big\langle \vec{z}, \overset{\in W}{\sum_{i=1}^{k} b_k \vec{v}_i} \Big\rangle = \sum_{i=1}^{k} \overline{b_k} \cancelto{0\,=\,0}{\langle \vec{z}, \vec{v}_i \rangle} = 0$$

$$\therefore \quad \vec{z} \in W^{\perp}$$

For uniqueness, suppose $\exists\, \vec{u}\,' \in W$ and $\vec{z}\,' \in W^{\perp}$ such that:

$$\vec{y} = \underset{\substack{\uparrow \\ W}}{\vec{u}} + \underset{\substack{\uparrow \\ W^{\perp}}}{\vec{z}} = \underset{\substack{\uparrow \\ W}}{\vec{u}\,'} + \underset{\substack{\uparrow \\ W^{\perp}}}{\vec{z}\,'} \Rightarrow \underset{\substack{\uparrow \\ W}}{\vec{u} - \vec{u}\,'} = \underset{\substack{\uparrow \\ W^{\perp}}}{\vec{z}\,' - \vec{z}} \in W \cap W^{\perp}$$

<u>Claim</u>: $W \cap W^{\perp} = \{\vec{0}\}$

<u>Pf</u>: Take $\vec{w} \in W \cap W^{\perp}$. Then: $\langle \underset{\substack{\uparrow \\ W}}{\vec{w}}, \underset{\substack{\uparrow \\ W^{\perp}}}{\vec{w}} \rangle = 0$

$$\Longleftrightarrow \vec{w} = \vec{0}$$

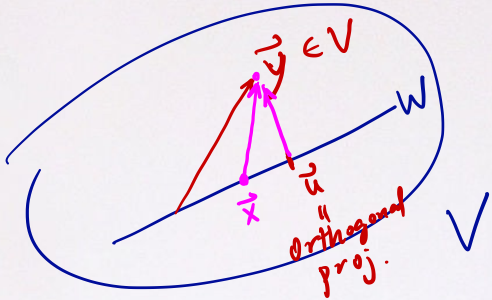
This implies: $\vec{u} - \vec{u}\,' = \vec{z}\,' - \vec{z} = \vec{0}$

$$\Longleftrightarrow \vec{u} = \vec{u}\,' \text{ and } \vec{z} = \vec{z}\,'.$$

Corollary: With notations as above, then:

$$\|\underset{\underset{V}{\in}}{\vec{y}} - \underset{\underset{W}{\in}}{\vec{x}}\| \geq \|\underset{\underset{V}{\in}}{\vec{y}} - \underset{\underset{W}{\in}}{\vec{u}}\| \quad \text{for } \forall \vec{x} \in W$$

and equality holds iff $\vec{x} = \vec{u}$



Remark: Orthogonal projection is the vector in W closest to $\vec{y}$.

Proof: Let $\vec{x} \in W$. Then: $\underset{W}{\vec{y} = \vec{u}} + \underset{W^{\perp}}{\vec{z}} \Rightarrow \vec{z} = \vec{y} - \vec{u}$

$$\|\vec{y} - \vec{x}\|^2 = \|\vec{u} + \vec{z} - \vec{x}\|^2 = \langle (\underset{W}{\vec{u}} - \underset{W}{\vec{x}}) + \underset{W^{\perp}}{\vec{z}},\ (\underset{W}{\vec{u}} - \underset{W}{\vec{x}}) + \underset{W^{\perp}}{\vec{z}} \rangle$$

$$= \langle \vec{u} - \vec{x}, \vec{u} - \vec{x} \rangle + \underbrace{\langle \underset{W}{\vec{u} - \vec{x}}, \underset{W^{\perp}}{\vec{z}} \rangle}_{= 0} + \underbrace{\langle \vec{z}, \vec{u} - \vec{x} \rangle}_{= 0} + \langle \vec{z}, \vec{z} \rangle$$

$$= \|\vec{u} - \vec{x}\|^2 + \|\vec{z}\|^2 \geq \|\vec{z}\|^2 = \|\vec{y} - \vec{u}\|^2$$

The equality holds iff $\|\vec{u} - \vec{x}\|^2 = 0$ iff $\vec{u} = \vec{x}$.