Lecture 17: Recall:

Definition: Let V_1, V_2 be vector spaces over F. A pair (Y, μ), where Y is a vector space over F and $\mu : V_1 \times V_2 \to Y$ is a linear map, is called the tensor product of V_1 and V_2 if the following condition holds:

(*) whenever β_1 is a basis for V_1 and β_2 is a basis for V_2, then:

$$\mu(\beta_1 \times \beta_2) \overset{\text{def}}{=} \{ \mu(\vec{x}_1, \vec{x}_2) : \vec{x}_1 \in \beta_1, \vec{x}_2 \in \beta_2 \}$$ is a basis for Y.

Remark: We write $V_1 \otimes V_2$ for Y.

We write $\vec{x}_1 \otimes \vec{x}_2$ for $\mu(\vec{x}_1, \vec{x}_2)$.

Theorem: Let Y be a vector space and $\mu : V_1 \times V_2 \to Y$ be a linear map. Suppose there exist bases γ_1 for V_1 and γ_2 for V_2 such that $\mu(\gamma_1 \times \gamma_2)$ is a basis for Y. Then $(*)$ holds for any choice of basis.

<u>Remark:</u> • In general,

$$V \otimes W \stackrel{\text{def}}{=} \left\{ \sum_i a_i \vec{v}_i \otimes \vec{w}_i \ : \ a_i \in F, \ \vec{v}_i \in V, \ \vec{w}_i \in W \right\}.$$

• Let $V_1, V_2, \ldots, V_k$ be vector space over a field F.

A pair (Y, μ), where Y is a vector space over F and $\mu : V_1 \times V_2 \times \ldots \times V_k \to Y$ is a k-linear map (i.e. for any i and for any $\vec{v}_1 \in V_1, \ldots, \vec{v}_{i-1} \in V_{i-1}, \vec{v}_{i+1} \in V_{i+1}, \ldots, \vec{v}_k \in V_k$, the map $\mu(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{i-1}, \cdot, \vec{v}_{i+1}, \ldots, \vec{v}_k)$ is linear), is called the tensor product of $V_1, V_2, \ldots, V_k$ if:

(*) whenever β_i is a basis for V_i $(i = 1, 2, \ldots, k)$,

$$\{ \mu(\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k) : \vec{x}_i \in \beta_i \} \text{ is a basis for } Y.$$

We write $V_1 \otimes V_2 \otimes \cdots \otimes V_k$ for Y

and $\vec{x}_1 \otimes \vec{x}_2 \otimes \cdots \otimes \vec{x}_k$ for $M(\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k)$.

<u>Example</u>: · Let $V = P(F)$. Then: $V \otimes V = P(F^2)$ = space of polynomials in two variables under the product defined to be $f(x) \otimes g(x) = f(x_1) \cdot g(x_2)$.

(∵ let $\beta = \{1, x, x^2, \ldots\}$ be a basis for V.

Then: $\beta \otimes \beta = \{ x^i \otimes x^j : i, j = 0, 1, 2, \ldots \}$

$$= \{ x_1^i x_2^j : i, j = 0, 1, 2, \ldots \}$$

is a basis for $P(F^2)$.)

- If V is any vector space over F, then under $\otimes$ as scalar multiplication $(\vec{v} \otimes a = a\vec{v})$,

$$V \otimes F = V$$

- Let $V = \{(x_1, x_2, \ldots, x_n) : x_i \in F\}$

$$W = \left\{ \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} : y_i \in F \right\}.$$

Under the product $\otimes$ defined to be $\vec{v} \otimes \vec{w} = \vec{w}\,\vec{v}$,

$$V \otimes W = M_{m \times n}(F).$$

Theorem: Let V, W be finite dimensional vector spaces over F. Let V^* be the dual space of V. Then:

$$V^* \otimes W = \mathcal{L}(V, W) = \text{space of linear transf. from } V \text{ to } W.$$

(with multiplication defined as $f \otimes \vec{w} : V \to W$ given by:

$$\underset{W}{f \otimes \vec{w}}\,(\underset{V}{\vec{v}}) = \underset{F}{f(\vec{v})} \cdot \underset{W}{\vec{w}} \in W.$$

Proof: Let $\alpha = \{f_1, f_2, \ldots, f_n\}$ be the dual basis for V^* of basis $\beta = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ for V.

Let $\gamma = \{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_m\}$ be a basis for W.

Then: consider $\Psi = \{ f_i \otimes \vec{w}_j : f_i \in \alpha, \vec{w}_j \in \gamma \}$.

Ψ spans $\mathcal{L}(V,W)$: For any $T \in \mathcal{L}(V,W)$, T can be determined by:

$$(*) \quad T(\vec{v}_j) = \sum_{i=1}^{m} a_{ij} \vec{w}_i \quad \text{for } j = 1, 2, \ldots, n.$$

Let $\tilde{T} = \sum_{j=1}^{n} \sum_{i=1}^{m} a_{ij} f_j \otimes \vec{w}_i$

Then: $\tilde{T}(\vec{v}_j) = T(\vec{v}_j) \Rightarrow T = \tilde{T} = \underbrace{\sum_{j=1}^{n}\sum_{i=1}^{m} a_{ij} f_j \otimes \vec{w}_i}_{\in \, \mathrm{Span}(\Psi)}$

Also, $|\Psi| = mn = \dim(\mathcal{L}(V,W))$

$\therefore$ Ψ is a basis!

Remark: If V and W are infinite-dimensional,

$$V^* \otimes W \overset{??}{=} \mathcal{L}(V, W) \; ??$$

No. In fact,

$$V^* \otimes W = \{ T \in \mathcal{L}(V,W) : \dim(R(T)) < \infty \}$$

$=$ set of finite rank linear transformations in $\mathcal{L}(V, W)$.

Note that: $f \otimes \vec{w}$ is at most rank 1.

$\therefore$ A linear combination of such transformations must have finite rank!

Trace

Consider $V^* \otimes V$. We can define a natural linear functional

$Tr : V^* \otimes V \to F$ as follows:

let $\beta = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ be a basis for V and
$\beta^* = \{f_1, f_2, \ldots, f_n\}$ be a basis for V^*.

Then: $\beta^* \otimes \beta = \{ f_i \otimes \vec{v}_j : 1 \le i, j \le n\}$ form a basis for $V^* \otimes V$.

Define: $Tr(f_i \otimes \vec{v}_j) = f_i(\vec{v}_j)$

Tr is called the trace on V

Example: Consider a linear operator $T: V \to V$ on V over F.

Let $A = [T]_\beta$. Then: $T(\vec{v}_j) = \sum_{i=1}^{n} A_{ij} \vec{v}_i$

In particular, $T = \sum_{i,j} A_{ij} \, f_j \otimes \vec{v}_i$

Then: $Tr(T) = \sum_{i,j} A_{ij} \, Tr(f_j \otimes \vec{v}_i)$

$= \sum_{i,j} A_{ij} \, f_j(\vec{v}_i) = \sum_{i,j} A_{ij} \delta_{ij}$

$= A_{11} + A_{22} + \dots + A_{nn}$

(Same definition as in Math 1038!!)

Inner product and norm

Assume $F = \mathbb{R}$ or $\mathbb{C}$.

Definition: Let V be a vector space over F. An inner product on V is a map $\langle \cdot, \cdot \rangle : V \times V \to F$ s.t. $\forall \vec{x}, \vec{y}, \vec{z} \in V$ and $c \in F$, it satisfies:

(a) $\langle \vec{x} + \vec{z}, \vec{y} \rangle = \langle \vec{x}, \vec{y} \rangle + \langle \vec{z}, \vec{y} \rangle$

(b) $\langle c\vec{x}, \vec{y} \rangle = c \langle \vec{x}, \vec{y} \rangle$

(c) $\overline{\langle \vec{x}, \vec{y} \rangle} = \langle \vec{y}, \vec{x} \rangle$

(d) $\langle \vec{x}, \vec{x} \rangle > 0$ if $\vec{x} \neq \vec{0}$

$\in \mathbb{R}$

Remark: • (a), (b) say that the inner product is linear in its argument.

• If $F = \mathbb{R}$, $\langle \vec{x}, \vec{y} \rangle = \langle \vec{y}, \vec{x} \rangle$

Example: • For $\vec{x} = (a_1, a_2, \dots, a_n)$, $\vec{y} = (b_1, b_2, \dots, b_n) \in F^n$
($F = \mathbb{R}, \mathbb{C}$)

We have: standard inner product

$$\langle \vec{x}, \vec{y} \rangle \overset{\text{def}}{:=} \sum_{i=1}^{n} a_i \overline{b_i}$$

• If $\langle \cdot, \cdot \rangle$ is an inner product on V, and $r > 0$,

then: $\langle \vec{x}, \vec{y} \rangle' \overset{\text{def}}{:=} r \langle \vec{x}, \vec{y} \rangle$ is another inner product on V.

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- Let $V = C([0,1])$ be vector space of real-valued continuous functions on $[0,1]$. Then: for $f, g \in V$,

$$\langle f, g \rangle \overset{\text{def}}{=} \int_0^1 f(t)g(t)\,dt$$ defines an inner product on V.

$(F = \mathbb{R}, \mathbb{C})$

- Let $V = M_{n \times n}(F)$. For $A, B \in V$, we define:

sum of diagonal entries.

$$\langle A, B \rangle \overset{\text{def}}{=} \operatorname{tr}(B^* A)$$

where B^* is the conjugate transpose of B defined by:

$$B^* = \overline{B}^T$$

For $A, B, C \in V$ and $\lambda \in F$, we check:

(a) $\langle A+B, C\rangle = \mathrm{tr}(C^*(A+B)) = \mathrm{tr}(C^*A + C^*B)$

$= \mathrm{tr}(C^*A) + \mathrm{tr}(C^*B)$

$= \langle A, C\rangle + \langle B, C\rangle$

(b) $\langle \lambda A, B\rangle = \mathrm{tr}(B^*(\lambda A)) = \mathrm{tr}(\lambda(B^*A))$

$= \lambda\, \mathrm{tr}(B^*A)$

$= \lambda \langle A, B\rangle$

(c) $\overline{\langle A, B\rangle} = \overline{\mathrm{tr}(B^*A)} = \mathrm{tr}(\overline{\underbrace{B^*}_{=\overline{B}^T}A}) = \mathrm{tr}(B^T\overline{A})$

$= \mathrm{tr}((B^T\overline{A})^T) = \mathrm{tr}(\overline{A}^T\underbrace{(B^T)^T}_{B})$

$\mathrm{Tr}(C) = \mathrm{Tr}(C^T)$

$= \mathrm{tr}(A^*B) = \langle B, A\rangle.$

(d) $\langle A, A \rangle = tr(A^* A) = \sum_{i=1}^{n} (A^* A)_{ii}$

$$= \sum_{i=1}^{n} \Big(\sum_{k=1}^{n} (A^*)_{ik} A_{ki} \Big)$$

$$(A^*)_{ik} = \bar{A}^T_{ik} = \bar{A}_{ki}$$

$$= \sum_{i=1}^{n} \sum_{k=1}^{n} \underbrace{\bar{A}_{ki} A_{ki}}_{= |A_{ki}|^2}$$

$$\langle A, A \rangle = \sum_{i=1}^{n} \sum_{k=1}^{n} |A_{ki}|^2 \geq 0$$

and $\langle A, A \rangle = 0$ iff $A_{ki} = 0 \ \forall k, i$ (i.e. $A = 0$)

Definition: A vector space V equipped with an inner product is called an inner product space.

If $F = \mathbb{C}$, we call V a complex inner product space.

If $F = \mathbb{R}$, we call V a real inner product space.

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Proposition: Let V be an inner product space. Then, $\forall \vec{x}, \vec{y}, \vec{z} \in V$ and $\forall c \in F$, we have:

(a) $\langle \vec{x}, \vec{y} + \vec{z} \rangle = \langle \vec{x}, \vec{y} \rangle + \langle \vec{x}, \vec{z} \rangle$

(b) $\langle \vec{x}, c\vec{y} \rangle = \overline{c} \langle \vec{x}, \vec{y} \rangle$

(c) $\langle \vec{x}, \vec{0} \rangle = \langle \vec{0}, \vec{x} \rangle = 0$

(d) $\langle \vec{x}, \vec{x} \rangle = 0$ iff $\vec{x} = \vec{0}$

(e) If $\langle \vec{x}, \vec{y} \rangle = \langle \vec{x}, \vec{z} \rangle$ for $\forall \vec{x} \in V$, then $\vec{y} = \vec{z}$.

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Proof:

(a) $\langle \vec{x}, \vec{y}+\vec{z} \rangle = \overline{\langle \vec{y}+\vec{z}, \vec{x} \rangle}$

$= \overline{\langle \vec{y}, \vec{x} \rangle + \langle \vec{z}, \vec{x} \rangle}$

$= \overline{\langle \vec{y}, \vec{x} \rangle} + \overline{\langle \vec{z}, \vec{x} \rangle} = \langle \vec{x}, \vec{y} \rangle + \langle \vec{x}, \vec{z} \rangle$

(b) $\langle \vec{x}, c\vec{y} \rangle = \overline{\langle c\vec{y}, \vec{x} \rangle} = \overline{c\langle \vec{y}, \vec{x} \rangle} = \bar{c}\,\overline{\langle \vec{y}, \vec{x} \rangle}$

$= \bar{c}\,\langle \vec{x}, \vec{y} \rangle$

(c) $\langle \vec{x}, \vec{0} \rangle = \langle \vec{x}, \vec{0}+\vec{0} \rangle = \langle \vec{x}, \vec{0} \rangle + \langle \vec{x}, \vec{0} \rangle$

So, $\langle \vec{x}, \vec{0} \rangle = 0$. Similarly, $\langle \vec{0}, \vec{x} \rangle = 0$

(d) If $\vec{x} = \vec{0}$, then $\langle \vec{x}, \vec{x} \rangle = 0$ by (c)

If $\vec{x} \neq \vec{0}$, then $\langle \vec{x}, \vec{x} \rangle > 0$ by definition.

(e) If $\langle \vec{x}, \vec{y} \rangle = \langle \vec{x}, \vec{z} \rangle$ for all $\vec{x} \in V$.

then $\langle \vec{x}, \vec{y} - \vec{z} \rangle = 0 \quad \forall \vec{x} \in V$.

In particular, we can choose $\vec{x} = \vec{y} - \vec{z}$.

Then: $\langle \vec{y} - \vec{z}, \vec{y} - \vec{z} \rangle = 0 \Rightarrow \vec{y} - \vec{z} = \vec{0}$ (by (d))

$\Rightarrow \vec{y} = \vec{z}$.

<u>Remark</u>: (a) + (b) together say that the inner product is <u>conjugate linear</u> in the second argument.

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Definition: Let V be an inner product space. For $\vec{x} \in V$, we can define the length or norm of $\vec{x}$ by:

$$\|\vec{x}\| \overset{\text{def}}{:=} \sqrt{\langle \vec{x}, \vec{x} \rangle}$$

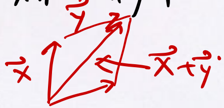
Proposition: Let V be an inner product space over F. Then, $\forall \vec{x}, \vec{y} \in V$ and $\forall c \in F$, we have:

(a) $\|c\vec{x}\| = |c| \cdot \|\vec{x}\|$

(b) $\|\vec{x}\| \geq 0$, and $\|\vec{x}\| = 0$ iff $\vec{x} = \vec{0}$.

(c) $|\langle \vec{x}, \vec{y} \rangle| \leq \|\vec{x}\| \, \|\vec{y}\|$ (Cauchy-Schwarz Inequality)

(d) $\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$ (Triangle inequality)



Proof: (a) $\| c\vec{x} \| = \sqrt{\langle c\vec{x}, c\vec{x} \rangle} = \sqrt{c\,\underbrace{\bar{c}}_{}\langle \vec{x}, \vec{x} \rangle}$

$c\bar{c} = |c|^2$

$= |c| \sqrt{\langle \vec{x}, \vec{x} \rangle} = |c|\,\|\vec{x}\|.$

(b) $\|\vec{x}\| = \sqrt{\langle \vec{x}, \vec{x} \rangle} \geq 0$ (by definition)

$\|\vec{x}\| = 0 \iff \langle \vec{x}, \vec{x} \rangle = 0$ iff $\vec{x} = \vec{0}$

(c) and (d) are shown in the tutorial.

Orthogonality

Definition: Let V be an inner product space. We say $\vec{x}, \vec{y} \in V$ are <u>orthogonal</u> (or perpendicular) if $\langle \vec{x}, \vec{y} \rangle = 0$.

A subset $S \subset V$ is called orthogonal if any two distinct vectors in S are orthogonal.

A <u>unit vector</u> in V is a vector $\vec{x} \in V$ with $\|\vec{x}\| = 1$.

A subset $S \subset V$ is called <u>orthonormal</u> if S is orthogonal and all vectors in S are unit vectors.

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e.g. Let H be the space of continuous complex-valued functions on $[0, 2\pi]$. We have inner product defined by:

$$\langle f, g \rangle \overset{\text{def}}{=} \frac{1}{2\pi} \int_0^{2\pi} f(t)\, \overline{g(t)}\, dt \quad \text{for } f, g \in H$$

For any $n \in \mathbb{Z}$ ← integer, let

$$f_n(t) = e^{int} \overset{\text{def}}{:=} \cos nt + i \sin nt \quad \text{for } t \in [0, 2\pi]$$

$\sqrt{-1}$ → i

and consider $S = \{ f_n : n \in \mathbb{Z} \} \subset H$

For any $m \neq n$, we have:

$$\langle f_m, f_n \rangle = \frac{1}{2\pi} \int_0^{2\pi} e^{imt}\, \overline{e^{int}}\, dt = \frac{1}{2\pi} \int_0^{2\pi} e^{i(m-n)t}\, dt = \frac{1}{2\pi} \left(\frac{1}{i(m-n)} \right) e^{i(m-n)t} \Big|_0^{2\pi}$$

$$\overline{e^{int}} = e^{-int}$$

$$\frac{1}{2\pi} \int_0^{2\pi} e^{i(m-n)t}\, dt \overset{\text{def}}{=} \frac{1}{2\pi} \int_0^{2\pi} \cos(m-n)t\, dt + i \frac{1}{2\pi} \int_0^{2\pi} \sin(m-n)t\, dt$$

$$= 0$$

Also, $\langle f_n, f_n \rangle = \frac{1}{2\pi}\int_0^{2\pi} e^{int}\,\underbrace{\overline{e^{int}}}_{= e^{-int}}\,dt = \frac{1}{2\pi}\int_0^{2\pi} 1\,dt = 1$

$\therefore$ S is orthonormal subset of H.

Definition: Let V be an inner product space. A subset of V is an **orthonormal basis** for V if it is an ordered basis which is orthonormal.

Proposition: Let V be an inner product space and $S = \{\vec{v}_1, \ldots, \vec{v}_k\}$ be an orthogonal subset of V consisting of non-zero vectors.

Then: $\forall\ \vec{y} \in \mathrm{Span}(S)$,

$$\vec{y} = \sum_{i=1}^{k} \left(\frac{\langle \vec{y}, \vec{v}_i \rangle}{\| \vec{v}_i \|^2} \right) \vec{v}_i$$

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Proof: Write $\vec{y} = \sum_{i=1}^{k} a_i \vec{v}_i$ for some $a_1, a_2, \ldots, a_k \in F$.

Take inner product with $\vec{v}_j$ on both sides gives:

$$\langle \vec{y}, \vec{v}_j \rangle = \sum_{i=1}^{k} a_i \langle \vec{v}_i, \vec{v}_j \rangle = a_j \|\vec{v}_j\|^2$$

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Corollary 1: If, in addition to above, S is orthonormal, then $\forall \vec{y} \in \mathrm{Span}(S)$,

$$\vec{y} = \sum_{i=1}^{k} \langle \vec{y}, \vec{v}_i \rangle \vec{v}_i$$

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<u>Corollary 2</u>: Let S be an orthogonal subset of an inner product space V consisting of non-zero vectors. Then, S is linearly independent.

<u>Proof</u>: If $\displaystyle\overbrace{\sum_{i=1}^{k} a_i \vec{v}_i}^{\in \operatorname{Span}(\{\vec{v}_1, \dots, \vec{v}_k\})} = \vec{0}$ for some $\vec{v}_1, \dots, \vec{v}_k \in S$ and $a_1, a_2, \dots, a_k \in F$,

By previous proposition,

$$a_i = \frac{\overset{0}{\langle \vec{0}, \vec{v}_i \rangle}}{\|\vec{v}_i\|^2} = 0 \quad \text{for } i = 1, 2, \dots, k. \quad //$$