Lecture 14:

Example: For $A = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix} \in M_{3\times 3}(\mathbb{R})$

$f_A(t) = -(t-4)(t-3)^2$ splits over $\mathbb{R}$.

$$\gamma_T(4) = \mu_T(4) = 1$$

But $\operatorname{rank}(\overbrace{A - 3I}^{B}) = \operatorname{rank}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 2$

$$(\underbrace{\text{Rank}(B)}_{2} + \underbrace{\text{Nullity}(B)}_{1} = 3)$$

$$\gamma_A(3) = 1 \neq \mu_A(3) = 2$$

$\therefore$ T is not diagonalizable.

Example: Consider $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ defined by:

$$T(f(x)) = f(1) + f'(0)x + (f'(0) + f''(0))x^2$$

Let $\beta = \{1, x, x^2\}$.

$$[T]_\beta = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix} \Rightarrow f_T(t) = -(t-1)^2(t-2)$$

splits over $\mathbb{R}$.

and the eigenvalues of T are 1 and 2.

$\therefore \gamma_T(2) = \mu_T(2) = 1$.

$$\text{Rank}([T]_\beta - I) = \text{rank}\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} = 1 \Rightarrow \gamma_T(1) = 2 = \mu_T(1)$$

$\therefore T$ is diagonalizable.

For $[T]_\beta$, the eigenspaces:

$$E_1 = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 : x_2 + x_3 = 0 \right\} = \text{span} \underbrace{\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}}_{\text{Basis}}$$

$$E_2 = \text{span} \left\{ \underbrace{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}}_{\text{Basis}} \right\}$$

$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$ is a basis of eigenvectors (of $[T]_\beta$) for $\mathbb{R}^3$.

$\therefore \{1, x - x^2, 1 + x^2\}$ is a basis of eigenvectors (of T) for $P_2(\mathbb{R})$.

Definition: Let T be a linear operator on a vector space V.

A subspace $W \subset V$ is called **T-invariant** if $T(W) \subseteq W$.

That is, $T(\vec{w}) \in W$ for $\forall \vec{w} \in W$.

Example: If T is a linear operator on V, then:

$\{\vec{0}\}$ is T-invariant

V is " "

$R(T)$ ✓ ✓

$N(T)$ ✓ ✓

E_λ ✓ ✓

eigenvalue

($\underset{T(\vec{v})}{\vec{w}} \in R(T)$, then: $T(T(\vec{v})) \in R(T)$)

($\vec{v} \in E_\lambda$, $T(\vec{v}) = \lambda\vec{v} \in E_\lambda$)

- For $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(a,b,c) = (a+b, b+c, 0)$

then x-y plane $\{(x,y,0) : x,y \in \mathbb{R}\}$ is T-invariant

x-axis $\{(x,0,0) : x \in \mathbb{R}\}$ is T-invariant

z-axis $\{(0,0,x) : x \in \mathbb{R}\}$ is NOT T-invariant.

$$T((0,0,\underset{\neq 0}{x})) = (0, \underset{\neq 0}{x}, 0) \notin \text{z-axis}$$

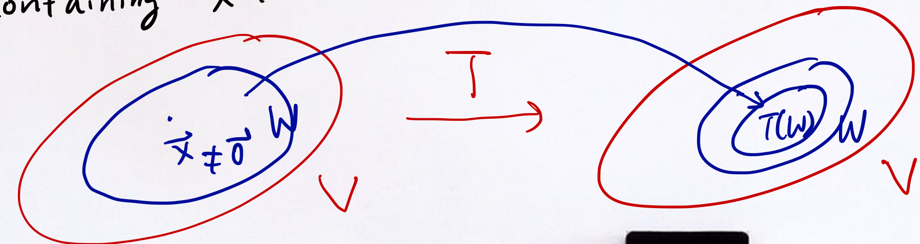
Def: Given a linear operator T on a vector space V, and a non-zero $\vec{x} \in V$, the subspace

$$W := \operatorname{span}\left(\{ T^k(\vec{x}) : k \in \mathbb{N} \}\right) \overset{\text{def}}{=} \operatorname{span}\left(\{ \vec{x}, T(\vec{x}), T^2(\vec{x}), \dots, T^k(\vec{x}), \dots \}\right)$$

$$\left(T^k \overset{\text{def}}{=} \underbrace{T \circ T \circ \dots \circ T}_{k \text{ times}} \right)$$

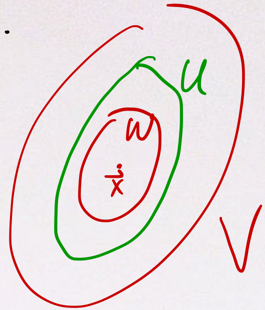
is called **T-cyclic subspace** of V generated by $\vec{x}$.

Prop: W is the smallest T-invariant subspace of V containing $\vec{x}$.



Proof: For any $\vec{w} \in W$, $\exists\, a_0, \ldots, a_k \in F$ s.t.

$$\vec{w} = \sum_{i=0}^{k} a_i T^i(\vec{x})$$



Then: $$T(\vec{w}) = \sum_{i=0}^{k} a_i T^{i+1}(\vec{x}) \in W.$$

$\therefore$ W is T-invariant.

If $U \subset V$ is a T-invariant subspace containing $\vec{x}$.

then: it also contains $\underset{\in U}{T(\vec{x})}$ and $\underset{\in U}{T^k(\vec{x})}$ by induction.

$\therefore$ $U \supset W$

<u>Example:</u>

- For $T: P(\mathbb{R}) \to P(\mathbb{R})$ defined by $T(f(x)) = f'(x)$ then T-cyclic subspace generated by x^n is:

$$\text{span}\left(\{x^n, nx^{n-1}, \ldots, n!\,x, n!\}\right) = P_n(\mathbb{R})$$

- Let $T: V \to V$ be linear. Then, a 1-dimensional T-invariant subspace $U \subset V$ is nothing but the span of an eigenvector of T.

[If U = 1-dim T-invariant subspace.
Then, $U = \text{span}\{\vec{v}\}$. Then: $T(\vec{v}) \in U$
$\Rightarrow T(\vec{v}) = \lambda \vec{v}$. $\therefore$ $\vec{v}$ = eigenvector of T.

Also, if $\vec{v} \in V$ is an eigenvector of T, then T-cyclic subspace generated by $\vec{v}$ is also $\text{span}\{\vec{v}\}$. $\left(= \{\vec{v}, \cancel{T(\vec{v})}_{\lambda\vec{v}}, \cancel{T^2(\vec{v})}_{\lambda^2\vec{v}} \ldots\}\right)$