# Lecture 13: Recall:

**Prop:** Let $T$ be a linear operator on a vector space $V$ and let $\lambda$ be an eigenvalue of $T$. Then, $\vec{v} \in V$ is an eigenvector of $T$ corresponding to $\lambda$ iff:

$$\vec{v} \in N(T - \lambda I_V) \setminus \{\vec{0}\}$$

**Def:** Let $T$ be a linear operator on a vector space $V$ and let $\lambda$ be an eigenvalue of $T$.

Then: the subspace $E_\lambda \stackrel{def}{:=} N(T - \lambda I_V) = \{\vec{x} \in V : T(\vec{x}) = \lambda \vec{x}\} \subset V$

is called the <u>eigenspace</u> of $T$ corresponding to $\lambda$.

Eigenspaces of a matrix $A \in M_{n\times n}(F)$ is defined as those of $L_A$

**Prop:** Let $T$ be a linear operator on a vector space $V$, and let $\lambda_1, \lambda_2, \dots, \lambda_k$ be distinct eigenvalues of $T$. If $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k$ are eigenvectors of $T$ corresponding to $\lambda_1, \lambda_2, \dots, \lambda_k$ respectively, then: $\{\vec{v}_1, \dots, \vec{v}_k\}$ are linearly independent.

<u>Corollary</u>: A linear operator $T$ on an $n$-dim vector space $V$ which has $n$ distinct eigenvalues is diagonalizable.

<u>Proof</u>: Let $\vec{v}_1, \dots, \vec{v}_n \in V$ be the eigenvectors corresponding to $n$ distinct eigenvalues. Then, the prop. says $\{\vec{v}_1, \dots, \vec{v}_n\}$ is lin. independent. $\therefore$ $\{\vec{v}_1, \dots, \vec{v}_n\}$ forms a basis of eigenvectors.

$\therefore$ $T$ is diagonalizable.

Def: Let $\lambda$ be an eigenvalue of a linear operator or matrix with characteristic polynomial $f(t)$. The algebraic multiplicity of $\lambda$, denoted $\mu_T(\lambda)$ or $\mu_A(\lambda)$ is the multiplicity of $\lambda$ as a zero of $f(t)$, i.e. the largest positive integer $k$ s.t. $(t-\lambda)^k \mid f(t)$.

Example: $\cdot$ 1 is eigenvalue of $I_V : \underset{\beta}{V} \to \underset{\beta}{V}$

with $\mu_{I_V}(1) = \dim(V)$

$$f(t) = \det\big(\underset{\underset{I_n}{\parallel}}{[I_V]_\beta} - t I_n\big) = \begin{pmatrix} 1-t & & & \\ & 1-t & & \\ & & \ddots & \\ & & & 1-t \end{pmatrix} = (1-t)^n$$

$\cdot$ $A = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 4 \\ 0 & 0 & 5 \end{pmatrix}$. $f_A(t) = (3-t)^2(5-t)$

$\mu_A(3) = 2$, $\mu_A(5) = 1$

**Prop:** Let $T$ be a linear operator on a finite-dim vector space $V$ and let $\lambda$ be an eigenvalue of $T$ with algebraic multiplicity $\mu_T(\lambda)$. Then:

$$1 \leq \dim(E_\lambda) \leq \mu_T(\lambda)$$

We call $\gamma_T(\lambda) \overset{\text{def}}{=} \dim(E_\lambda)$ the geometric multiplicity of $\lambda$.

**Proof:** Choose an ordered basis $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ for $E_\lambda$ and ($p = \dim(E_\lambda)$)

extend it to an ordered basis $\beta = \{\vec{v}_1, \ldots, \vec{v}_p, \vec{v}_{p+1}, \ldots, \vec{v}_n\}$ for $V$.

Then: $$[T]_\beta = \left( \Big| [\overset{=\lambda\vec{v}_1}{T(\vec{v}_1)}]_\beta \Big| \cdots \Big| [\overset{=\lambda\vec{v}_p}{T(\vec{v}_p)}]_\beta \Big| \cdots \right) = \begin{pmatrix} \lambda & 0 & & 0 & \\ 0 & \lambda & & \vdots & \\ 0 & 0 & \cdots & \lambda & * \\ \vdots & \vdots & & 0 & \\ 0 & 0 & & \vdots & \\ & & & 0 & \end{pmatrix}$$

$$= \left(\begin{array}{c|c} \lambda I_p & B \\ \hline O & C \end{array}\right)$$

$$\Rightarrow \quad f_T(t) = \det\left(\begin{array}{c|c} (\lambda - t) I_p & B \\ \hline 0 & C - t I_{n-p} \end{array}\right)$$

$$= \det\left((\lambda - t) I_p\right) \underline{\det\left(C - t I_{n-p}\right)}$$

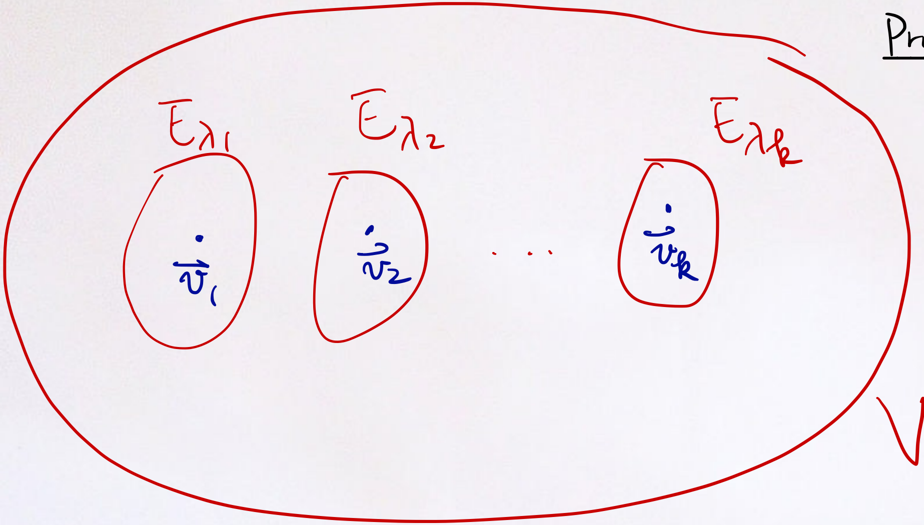
$$= (\lambda - t)^p \det\left(C - t I_{n-p}\right)$$

$$\therefore \quad (\lambda - t)^p \mid f_T(t)$$

$$\therefore \quad \mu_T(\lambda) \geq p = \gamma_T(\lambda)$$

**Lemma:** Let $T$ be a linear operator, and let $\lambda_1, \lambda_2, \dots, \lambda_k$ distinct eigenvalues of $T$. For each $i = 1, 2, \dots, k$, let $\vec{v}_i \in E_{\lambda_i}$.

If $\vec{v}_1 + \vec{v}_2 + \dots + \vec{v}_k = \vec{0}$, then $\vec{v}_i = \vec{0}$ for all $i$.




**Proof:** If not, say $\vec{v}_1, \dots, \vec{v}_s \neq \vec{0}$

then:

$$\underset{\neq \vec{0}}{\vec{v}_1} + \underset{\neq \vec{0}}{\vec{v}_2} + \dots + \underset{\neq \vec{0}}{\vec{v}_s} = \vec{0}$$

It contradicts to our previous proposition that $\vec{v}_1, \dots, \vec{v}_s$ must be lin. independent.

**Proposition:** Let $T$ be a linear operator, and let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be distinct eigenvalues of $T$. For each $i = 1, 2, \ldots, k$, let $S_i \subset E_{\lambda_i}$ be a finite linearly independent subset. Then:

$S = S_1 \cup S_2 \cup \cdots \cup S_k$ is a linearly independent subset of $V$.

**Proof:** Write $S_i = \{\vec{v}_{i1}, \vec{v}_{i2}, \ldots, \vec{v}_{in_i}\}$ for $i = 1, 2, \ldots, k$.

Suppose $\exists\, a_{ij} \in F$ for $1 \le j \le n_i$ and $1 \le i \le k$ such that

$$\sum_{i=1}^{k} \underbrace{\sum_{j=1}^{n_i} a_{ij} \vec{v}_{ij}}_{\overset{\text{def}}{\|}} = \vec{0}$$

$w_i \in E_{\lambda_i}$

Then: $\overset{E_{\lambda_1}}{\overset{\cup}{w_1}} + \overset{E_{\lambda_2}}{\overset{\cup}{w_2}} + \cdots + \underset{\in E_{\lambda_k}}{w_k} = \vec{0} \;\Rightarrow\; w_i = \sum_{j=1}^{n_i} a_{ij} \vec{v}_{ij} = \vec{0}$ for all $i$.

Then: $a_{ij} = 0$ for all $i$ and $j$

(for $S_i$ are lin. independent for all $i$.

$\therefore$ $S_1 \cup S_2 \cup \dots \cup S_k$ is linearly independent.

Theorem: Let $T$ be a linear operator on a finite dimensional vector space $V$ such that the characteristic polynomial $f_T(t)$ splits.

Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be distinct eigenvalues of $T$.

Then: (a) $T$ is diagonalizable iff: $\mu_T(\lambda_i) = \gamma_T(\lambda_i)$ for $i = 1, 2, \ldots, k$

(b) If $T$ is diagonalizable and $\beta_i$ is an ordered basis for $E_{\lambda_i}$ for each $i$, then: $\beta := \beta_1 \cup \beta_2 \cup \ldots \cup \beta_k$ is an ordered basis for $V$ consisting of eigenvectors.

(so that $[T]_\beta$ is a diagonal matrix)

**Proof:** Write $n = \dim(V)$, and $m_i = \mu_T(\lambda_i)$ and $d_i = \gamma_T(\lambda_i) = \dim(E_{\lambda_i})$ for all $i$.

Suppose $T$ is diagonalizable and $\beta$ is a basis for $V$ consisting of eigenvectors of $T$.

(e.g. $\beta = \{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4, \vec{v}_5, \ldots, \vec{v}_n\}$)


For each $i$, let $\beta_i = \beta \cap E_{\lambda_i}$ and $n_i \overset{\text{def}}{=} \#\beta_i$

Then: $n_i \leq d_i = \dim(E_{\lambda_i})$ ($\because$ $\beta_i$ is lin. independent)

Also, $d_i \leq m_i$ (last lecture)

So, we have $n_i \leq d_i \leq m_i$ for all $i$.

$$\therefore \quad n = \sum_{i=1}^{k} n_i \leq \sum_{i=1}^{k} d_i \leq \sum_{i=1}^{k} m_i = n = \dim(V)$$

$$\therefore \quad \sum_{i=1}^{k} d_i - \sum_{i=1}^{k} n_i = 0 \iff \sum_{i=1}^{k} \overset{\geq 0}{(d_i - n_i)} = 0$$

$$\Rightarrow \quad d_i = n_i \quad \text{for all } i.$$

$$\therefore \quad \sum_{i=1}^{k} m_i - \sum_{i=1}^{k} d_i = 0 \iff \sum_{i=1}^{k} \overset{\geq 0}{(m_i - d_i)} = 0$$

$$\Rightarrow \quad d_i = m_i \quad \text{for all } i.$$

$$\therefore \quad n_i = \overset{\dim(E_{\lambda_i})}{\overset{\shortparallel}{d_i}} = m_i \quad \text{for all } i$$

(So, $\beta_i$ is a basis of $E_{\lambda_i}$)

Conversely, suppose $m_i = d_i \ \forall i$.

For each $i$, let $\beta_i$ be the ordered basis of $E_{\lambda_i}$

and let $\beta = \beta_1 \cup \beta_2 \cup \cdots \cup \beta_k$.

Then: from previous proposition, we know $\beta$ is linearly independent.

But $\#\beta = \sum_{i=1}^{k} d_i = \sum_{i=1}^{k} m_i = n = \dim(V)$

$$\#\beta = |\beta_1| + |\beta_2| + \cdots + |\beta_k|$$

$$|\beta_1| = \dim(E_{\lambda_1}) = d_1, \quad |\beta_2| = \dim(E_{\lambda_2}) = d_2, \quad |\beta_k| = \dim(E_{\lambda_k}) = d_k$$

$\therefore$ $\beta$ is a basis for $V$ of eigenvectors

$\therefore$ $T$ is diagonalizable.

**Example:** Let $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ be defined by:

$$T(f(x)) = f(x) + (x+1) f'(x)$$

Then: $A := [T]_\beta = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}$

where $\beta = \{1, x, x^2\}$ = standard ordered basis for $P_2(\mathbb{R})$.

$\therefore$ the char. poly:

$$\det(A - tI_3) = \det\begin{pmatrix} 1-t & 1 & 0 \\ 0 & 2-t & 2 \\ 0 & 0 & 3-t \end{pmatrix} = (1-t)^1 (2-t)^1 (3-t)^1$$

$$\left.\begin{aligned} 1 \le \gamma_T(1) \le \mu_T(1) = 1 \\ 1 \le \gamma_T(2) \le \mu_T(2) = 1 \\ 1 \le \gamma_T(3) \le \mu_T(3) = 1 \end{aligned}\right\} \quad \begin{aligned} \gamma_T(1) = \mu_T(1) \\ \gamma_T(2) = \mu_T(2) \\ \gamma_T(3) = \mu_T(3) \end{aligned} \quad \Rightarrow \text{Diagonalizable}$$

$$\cancel{E_1} \quad N(\underbrace{\overset{=[T]_\beta}{A} - 1 I_3}_{[T - 1 I_V]_\beta}) = N\begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix} = \left\{ a\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} : a \in \mathbb{R} \right\} \subseteq \mathbb{R}^3$$

$$\Rightarrow \quad E_1 = N(T - 1 I_V) = \{ a1 : a \in \mathbb{R} \} \subseteq P_2(\mathbb{R})$$

Similarly, $N(A - 2I_3) = N\begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{pmatrix} = \left\{ a\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} : a \in \mathbb{R} \right\}$

$$E_2 = \{ a(1+x) : a \in \mathbb{R} \} \subset P_2(\mathbb{R})$$

$$E_3 = \{ a(\overbrace{1 + 2x + x^2}^{(1+x)^2}) : a \in \mathbb{R} \}$$

$\beta = \{ 1, 1+x, (1+x)^2 \}$ is a basis of eigenvectors for $V$.

$$\begin{tikzcd} \mathbb{R}^3 \arrow[r, "{[T]_\beta}"] \arrow[d, leftrightarrow] & \mathbb{R}^3 \arrow[d, leftrightarrow] \\ P_2(\mathbb{R}) \arrow[r, "T"] & P_2(\mathbb{R}) \end{tikzcd}$$