Lecture 12:

Def: Let T be a linear operator on a vector space V/F.
A non-zero vector $\vec{v} \in V$ is called an eigenvector of T if $\exists\, \lambda \in F$ s.t. $T(\vec{v}) = \lambda \vec{v}$. In this case, $\lambda \in F$ is called an eigenvalue corresponding to the eigenvector $\vec{v}$.

For a square matrix $A \in M_{n \times n}(F)$, a non-zero vector $\vec{v} \in F^n$ is called an eigenvector of A if it is an eigenvector of L_A.

That is: $A\vec{v} = \lambda \vec{v}$ for some $\lambda \in F$.

λ is called the eigenvalue corresponding to the eigenvector $\vec{v}$.

Prop: A linear operator $T: V \to V$ ($V =$ fin-dim) is diagonalizable iff $\exists$ an ordered basis β for V consisting of eigenvectors of T.

In such case, if $\beta = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$, then:

$$[T]_\beta = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}$$

where λ_j is the eigenvalue of T corresponding to $\vec{v}_j$.

Prop: Let $A \in M_{n\times n}(F)$. Then $\lambda \in F$ is an eigenvalue of A iff $\det(A - \lambda I_n) = 0$.

Pf: $\lambda \in F$ is an eigenvalue of A

$\iff \exists \vec{v} \in F^n \setminus \{\vec{0}\}$ s.t. $A\vec{v} = \lambda \vec{v}$.

$\iff (A - \lambda I_n)\vec{v} = \vec{0}$

$\iff A - \lambda I_n$ is singular

$\iff A - \lambda I_n$ is not invertible

$\iff \det(A - \lambda I_n) = 0$.

Def: The characteristic polynomial of $A \in M_{n\times n}(F)$. is defined as the polynomial $f_A(t) \overset{def}{=} \det(A - tI_n) \in P_n(F)$

Def: Let T be a linear operator on an n-dim vector space V. Choose an ordered basis β for V. Then, the characteristic polynomial of T is defined as the characteristic polynomial of $[T]_\beta$.
(i.e. $f_T(t) \overset{def}{=} \det([T]_\beta - tI_n) \in P_n(F)$)

Prop: $f_T(t)$ is well-defined, i.e. independent of the choice of β.

Pf: If β' is another ordered basis for V, then:

$$[T]_{\beta'} = Q^{-1}[T]_{\beta}Q \quad \left(Q = [I_V]_{\beta'}^{\beta} \right)$$

Then:

$$\begin{aligned} \det([T]_{\beta'} - tI_n) &= \det(Q^{-1}[T]_{\beta}Q - tI_n) \\ &= \det(Q^{-1}([T]_{\beta} - tI_n)Q) \\ &= \cancel{\underbrace{\det(Q^{-1})}_{\frac{1}{\det(Q)}}}\det([T]_{\beta} - tI_n)\cancel{\det(Q)} \\ &= f_T(t). \end{aligned}$$

Prop: Let $A \in M_{n\times n}(F)$. Then:

- $f_A(t)$ is of degree n and with leading coefficient $(-1)^n$.
- A has at most n distinct eigenvalues.

Pf: Exercise

Def: A polynomial $f(t) \in P(F)$ splits over F if $\exists$ $c, a_1, a_2, \ldots, a_n \in F$ s.t. $f(t) = c(t-a_1)(t-a_2)\cdots(t-a_n)$

Prop: The characteristic polynomial of a diagonalizable linear operator on a finite-dim vector space V/F splits over F.

Pf: If V is a n-dim and $T: V \to V$ is diagonalizable,

then $\exists$ a basis $\beta \subset V$ s.t.

$$[T]_\beta = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}$$

Then: $f_T(t) = \det([T]_\beta - t I_n)$

$$= (-1)^n (t-\lambda_1)(t-\lambda_2) \cdots (t-\lambda_n)$$

Prop: Let T be a linear operator on a vector space V and let λ be an eigenvalue of T. Then, $\vec{v} \in V$ is an eigenvector of T corresponding to λ iff:

$$\vec{v} \in N(T - \lambda I_V) \setminus \{\vec{0}\}$$

Pf: Exercise.

$$T\vec{v} = \lambda \vec{v}$$

$$\Leftrightarrow (T - \lambda I_V)\vec{v} = \vec{0}$$

Def: Let T be a linear operator on a vector space V and let λ be an eigenvalue of T.

Then: the subspace $E_\lambda \stackrel{def}{:=} N(T - \lambda I_V) = \{\vec{x} \in V : T(\vec{x}) = \lambda \vec{x}\} \subset V$

is called the <u>eigenspace</u> of T corresponding to λ.

Eigenspaces of a matrix $A \in M_{n\times n}(F)$ is defined as those of L_A

Prop: Let T be a linear operator on a vector space V, and let $\lambda_1, \lambda_2, \dots, \lambda_k$ be distinct eigenvalues of T.
If $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k$ are eigenvectors of T corresponding to $\lambda_1, \lambda_2, \dots, \lambda_k$ respectively, then: $\{\vec{v}_1, \dots, \vec{v}_k\}$ are linearly independent.

Proof: We prove by induction on k.

For $k=1$, $\vec{v}_1 \neq \vec{0} \Rightarrow \{\vec{v}_1\}$ is lin. independent.

Suppose the statement holds for $k \geq 1$ distinct eigenvalues.

Let $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k, \vec{v}_{k+1}$ be eigenvectors corresponding to $k+1$ distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_k, \lambda_{k+1}$ of T.

If $a_1\vec{v}_1 + a_2\vec{v}_2 + \dots + a_k\vec{v}_k + a_{k+1}\vec{v}_{k+1} = \vec{0}$ for $a_i \in F$,

$\hat{N}(T-\lambda_{k+1}I_V)\setminus\{\vec{0}\}$

then applying $T-\lambda_{k+1}I_V$ to both sides gives:

$$a_1(\lambda_1-\lambda_{k+1})\vec{v}_1 + \dots + a_k(\lambda_k-\lambda_{k+1})\vec{v}_k = \vec{0}$$

By induction hypothesis,

$$a_1(\lambda_1 \overset{\neq 0}{-} \lambda_{k+1}) = \dots = a_k(\lambda_k \overset{\neq 0}{-} \lambda_{k+1}) = 0$$

$$\Rightarrow a_1 = a_2 = \dots = a_k = 0$$

$$\Rightarrow a_{k+1}\overset{\neq \vec{0}}{\vec{v}_{k+1}} = \vec{0}$$

$$\Rightarrow a_{k+1} = 0$$

$\therefore \{\vec{v}_1, \dots, \vec{v}_{k+1}\}$ is lin. indep.