2022-23 MATH2048: Honours Linear Algebra II Homework 9

Due: Need not to submit

These are exercises related to tensor product.

1. Consider $\mathbb{Q}^n$ and $\mathbb{R}$ as vectors spaces over $\mathbb{Q}$ (the field of rational numbers). Consider a bilinear form $\mu : \mathbb{Q}^n \times \mathbb{R} \to \mathbb{R}^n$ as follows: $\mu(\mathbf{x}, r) = r\mathbf{x}$ for any $\mathbf{x} \in \mathbb{Q}^n$ and $r \in \mathbb{R}$ (standard scalar multiplication). Let $\beta = \{\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_n\}$ be the standard ordered basis for $\mathbb{Q}^n$ over the field $\mathbb{Q}$. Suppose γ is a basis for $\mathbb{R}$ over the field $\mathbb{Q}$ (which is infinite dimensional). Prove that $(\mathbb{R}^n, \mu)$ forms a tensor product space. In other words, prove that $\mathbb{Q}^n \otimes \mathbb{R} = \mathbb{R}^n$.

2. The Kronecker product $\mu(A, B)$ of $A = (a_{ij}) \in M_{m\times n}$ and $B = (b_{ij}) \in M_{p\times q}$ is defined as:
$$\mu(A,B) = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{bmatrix} \in M_{mp\times nq}$$
Is the Kronecker product is a tensor product? Justify your answer.

3. Let V and W be two vector spaces over F (They may not be finite-dimensional). Let α be a basis for V, β be a basis for W, and γ be the dual basis of α.

   Then $V^* \otimes W = span(\{f \otimes w : f \in \gamma, w \in \beta\})$

   Consider the following linear map:
$$\Phi : V^* \otimes W \to \mathcal{L}(V, W)$$
   which is defined by
$$\Phi(f \otimes w)(v) = f(v)w \text{ for any } v \in V$$
   (a) Show that Φ is one-to-one.

   (b) Show that if V is finite-dimensional, then Φ is an isomorphism.

   Remark:

   1. In the lecture 17, we assume V and W are finite-dimensional and define $V^* \otimes W$ to be $V^* \otimes W = \mathcal{L}(V, W)$. That is, for any $f \in V^*$ and $w \in w$, $f \otimes w$ is defined by
$$\begin{aligned} f \otimes w : V &\to W \\ v &\mapsto f(v) \cdot w \end{aligned}$$
   The operation $f \otimes w$ is also proved to be a tensor product in the lecture.

   2. In this question, V and W may not be finite-dimensional. Besides, we allow $V^* \otimes W$ to be any tensor product of V^* and W instead of defining it specifically as in the lecture. By considering Φ, we show that as long as V is finite-dimensional, Φ is an isomorphism. Thus, the definition of $V^*\otimes W$ in the lecture is also a well-defined tensor product of V^* and W when V is finite-dimensional.