Recall

Reflexive spaces

Let X be a Banach space and $Q\colon X \to X^{**}$ be the *canonical map* (*natural embedding*), i.e.,

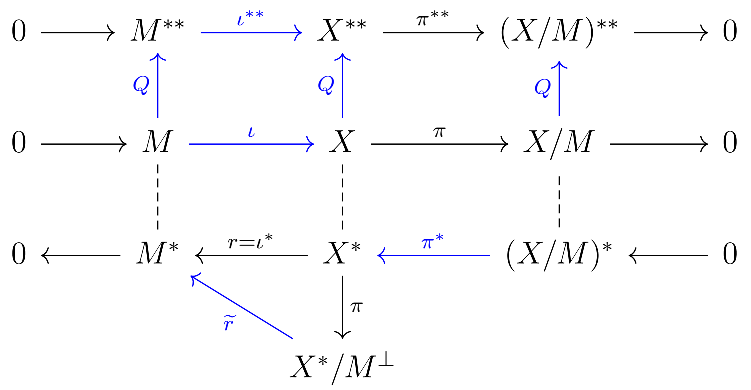
$$(Qx)(x^*) := x^*(x) \text{ or symmetrically, } \langle x^*, Qx\rangle := \langle x, x^*\rangle.$$

If $QX = X^{**}$, then X is called *reflexive.*

Let M be a closed subspace of a Banach space X. Recall the *annihilator*

$$M^{\perp} := \{x^* \in X^*\colon x^*(m) = 0, \forall\, m \in M\}.$$

By abuse of notation on Q (canonical maps), π (projections for quotient spaces) and ι (inclusions for subspaces), we may have the following (commutative) diagram:



where the blue arrows are isometries (which are not necessarily surjective). The dashed lines do not indicate maps. Note that

$$M^* = X^*/M^{\perp} \text{ by } \widetilde{r} \quad \text{and} \quad M^{\perp} = (X/M)^* \text{ by } \pi^*,$$

and so

$$(X/M)^{**} = (M^{\perp})^* = X^{**}/(M^{\perp})^{\perp} \quad \text{also} \quad \iota^{**}M^{**} = (M^{\perp})^{\perp}.$$

(i) $\{ X \text{ reflexive } \} \iff \{ X^* \text{ reflexive } \}$.

(ii) $\{ X \text{ reflexive } \} \iff \{ M \ \&\ X/M \text{ reflexive } \}$.

Remark. In another language, the rows of the above diagram are *short exact sequences.* Then (ii) follows quickly from the *short five lemma* with abstract diagram chasing.

If X is a **separable** Banach space, then:

- (Helley's selection) any bounded sequence in X^* has w^*-convergent subsequence.
- $\{$ In X^*, a sequence is w^*-convergent $\implies$ norm convergent. $\} \iff \{ \dim X < \infty \}$.
- $\{ X \text{ is reflexive } \} \implies \{$ any bounded sequence in X has weakly convergent subsequence $\}$.

Minkowski functional

Definition 1. Let A be a subset of a normed space (or topological vector space) X. The associated *Minkowski functional* $\mu_A \colon X \to [0,\infty]$ is defined by

$$\mu_A(x) := \inf\{t > 0 \colon x \in tA\} \tag{1}$$

for all $x \in X$, with the convention $\inf \emptyset = \infty$.

The property of A affects the behavior of μ_A. Here comes a natural question that when will μ_A become a norm on X.

Proposition 2. *Let μ_A be a Minkowski functional defined in (1). Then*

(1) (finiteness) $\{\ \mu_A(x) < \infty$ for all $x \in X\ \} \impliedby \{\ 0$ is an interior point of $A\ \}$.

(2) (subadditive) $\{\ \mu_A(x+y) \le \mu_A(x) + \mu_A(y)$ for $x, y \in X\ \} \impliedby \{\ A$ is convex $\}$.

(3) (positively homogeneous) Assume $0 \in A$. Then $\{\ \mu_A(\alpha x) = \alpha \mu_A(x)$ for $\alpha \ge 0$ and $x \in X\ \}$ always hold.

(a) ($\mathbb{R}$-absolutely homogeneous)
$\{\ \mu_A(\alpha x) = |\alpha| \mu_A(x)$ for $\alpha \in \mathbb{R}$ and $x \in X\ \} \impliedby \{\ A = -A\ \}$.

(b) ($\mathbb{C}$-absolutely homogeneous)
$\{\ \mu_A(\alpha x) = |\alpha| \mu_A(x)$ for $\alpha \in \mathbb{C}$ and $x \in X\ \} \impliedby \{\ A = e^{i\theta} A$ for all $\theta \in \mathbb{R}\ \}$.

(4) (positive definiteness) $\{$ if $\mu_A(x) = 0$, then $x = 0\ \} \impliedby \{\ A$ is bounded $\}$

Proof. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

(1) Let $x \in X$. Let V be an open neighborhood of 0 with $V \subset A$. Since the scalar product $\cdot : \mathbb{K} \times X \to X$ is continuous and $0 \cdot x = 0$, there exist $\delta > 0$ and an open neighborhood U of x such that

$$B(0,\delta) \cdot U \subset V$$

where $B(0,\delta)$ denotes the open ball of 0 with radius δ in $\mathbb{K}$. Then taking some $t \in (0,\delta)$, we have $t \cdot x \in V \subset A$, that is $x \in (1/t)A$, and so $\mu_A(x) \le 1/t < \infty$ by (1).

(2) Let $x, y \in X$. If $\mu_A(x) = \infty$ or $\mu_A(y) = \infty$, then the subadditivity holds trivially. Below we assume $\mu_A(x), \mu_A(y) < \infty$.

Let $\varepsilon > 0$. It follows from (1) that there exist $0 < \alpha \le \mu_A(x) + \varepsilon$ and $0 < \beta \le \mu_A(x) + \varepsilon$ such that $x \in \alpha A$ and $y \in \beta A$, that is, $x/\alpha, y/\beta \in A$. By the convexity of A,

$$\frac{x+y}{\alpha+\beta} = \frac{\alpha}{\alpha+\beta} \cdot \frac{x}{\alpha} + \frac{\beta}{\alpha+\beta} \cdot \frac{y}{\beta} \in A.$$

This shows that $x + y \in (\alpha+\beta)A$, then $\mu_A(x+y) \le \alpha + \beta \le \mu_A(x) + \mu_A(y) + 2\varepsilon$. The proof is completed by letting $\varepsilon \to 0$.

(3) By the assumption that $0 \in A$, we have $\mu_A(0 \cdot x) = \mu_A(0) = 0 \cdot \mu_A(x) = 0$ (with the convention $0 \cdot \infty = 0$). Let $\alpha > 0$. Then it is directly checked that

$$\{t > 0 \colon \alpha x \in tA\} = \alpha\{t > 0 \colon x \in tA\}.$$

Thus $\mu_A(\alpha x) = \alpha\mu_A(x)$ by taking infimum.

(a) It follows from $A = -A$ that

$$\{t > 0 \colon x \in tA\} = \{t > 0 \colon x \in t(-A)\} = \{t > 0 \colon -x \in tA\}.$$

Thus $\mu_A(x) = \mu_A(-x)$ by taking infimum. By (3) it suffices to check for $\alpha < 0$. In that case

$$\mu_A(\alpha x) = \mu_A((-\alpha)(-x)) = (-\alpha)\mu_A(-x) = |\alpha|\mu_A(x).$$

(b) Let $\theta \in \mathbb{R}$. It follows from $A = e^{i\theta}A$ that

$$\{t > 0 \colon x \in tA\} = \{t > 0 \colon x \in t(e^{-i\theta}A)\} = \{t > 0 \colon e^{i\theta}x \in tA\}.$$

Thus $\mu_A(x) = \mu_A(e^{i\theta}x)$ by taking infimum. Let $\alpha = |\alpha|e^{i\theta} \in \mathbb{C}$. By (3),

$$\mu_A(\alpha x) = \mu_A(|\alpha|e^{i\theta}x) = |\alpha|\mu_A(e^{i\theta}x) = |\alpha|\mu_A(x).$$

(4) Let $x \in X \setminus \{0\}$. Then (by the separation of vector space topology) there exists an open neighborhood V of 0 such that $x \notin V$. Since A is bounded, there exists $s > 0$ such that $A \subset tV$ for all $t > s$. This implies $x \notin (1/t)A$ for all $t > s$ since $x \notin V$. Then

$$\{\tau > 0 \colon x \in \tau A\} \subset [1/s, +\infty).$$

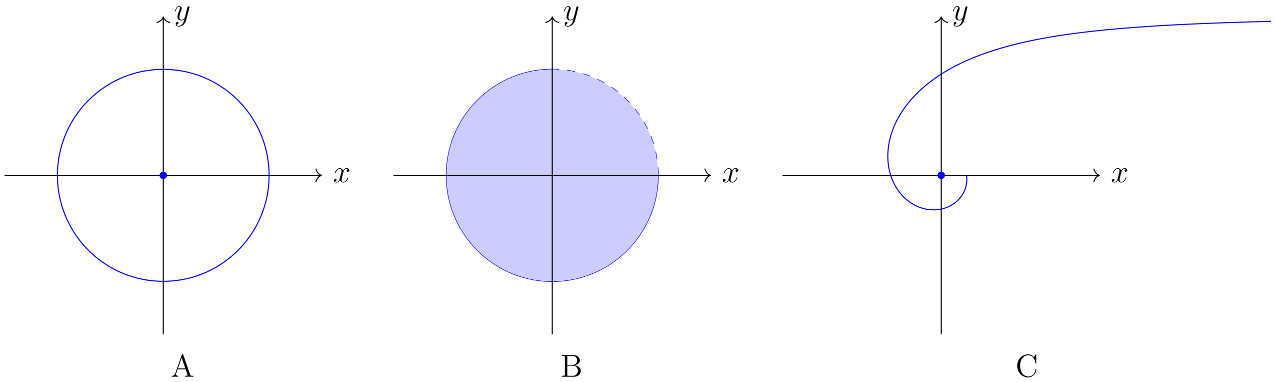
Hence $\mu_A(x) \geq 1/s > 0$ by (1).

□

It turns out that the above conditions are sufficient but not necessary. Counterexamples can be found in the plane to show that all the inverse directions " $\Longrightarrow$ " are false.

Example 3. Consider $X = \mathbb{C}$ and denote the absolute value of $x \in \mathbb{C}$ by $|x|$.

- Let $A = \{x \in \mathbb{C} \colon |x| = 1\} \bigcup \{0\}$. Then $\mu_A(x) = |x|$.
- Let $B = \{x \in \mathbb{C} \colon |x| \leq 1\} \setminus \{e^{i\theta} \colon \theta \in (0, \pi/2)\}$. Then $\mu_B(x) = |x|$.
- Let $C = \{e^{i\theta}/\theta \colon \theta \in (0, 2\pi]\} \bigcup \{0\}$. Then $\mu_C(x) = \theta(x)|x|$ if $x = |x|e^{i\theta(x)}$ and $\theta(x) \in (0, 2\pi]$.



Set A is for (1) and (2); Set B is for (a) and (b); Set C is for (4).