

## Recall

### Hahn-Banach Theorem(s)

*Dominated extension.* Let  $Y$  be a subspace of a vector space  $X$ . Let  $p$  be a positive homogeneous subadditive function on  $X$ . For every linear functional  $f \in Y^\#$  with  $f \leq p$  on  $Y$ , there exists  $F \in X^\#$  extending  $f$  and  $F \leq p$  on  $X$ .

*Continuous extension.* Let  $Y$  be a subspace of a normed space  $X$ . For every  $f \in Y^*$ , there exists  $F \in X^*$  extending  $f$  such that  $\|F\| = \|f\|$ .

*Existence of norm-attaining functional.* For every  $x_0$  in a normed space  $X$ , there exists  $f \in X^*$  such that  $\|f\| = 1$  and  $f(x_0) = \|x_0\|$ .

*Closure point checking.* Let  $Y$  be a subspace of a normed space  $X$ . Then  $x \in \bar{Y}$  if and only if for every  $f \in X^*$  with  $f = 0$  on  $Y$ , we have  $f(x) = 0$ .

*Hyperplane separation.* Let  $C$  be a closed convex subset of a normed space  $X$  and  $x_0 \in X \setminus C$ . Then there exists  $f \in X^*$  such that  $\sup_{y \in C} f(y) < f(x_0)$ .

If the dual space  $X^*$  is separable, then  $X$  is separable.

Recall that to apply *dominated extension* in the proof of *hyperplane separation*, we have introduced the *Minkowski functional*  $\mu_A$  defined for a set  $A$ . The properties of  $A$  determine the behavior of  $\mu_A$ . The way of defining Minkowski functional is useful to construct natural functions from sets and reveals properties of the space.

Let  $X, Y$  be normed spaces and  $T \in B(X, Y)$ . The adjoint operator  $T^*: Y^* \rightarrow X^*$  is (formally) defined as, for  $y^* \in Y^*, x \in X$ ,

$$T^*y^*(x) := y^*(Tx).$$

Then  $T^* \in B(Y^*, X^*)$  and  $\|T^*\| = \|T\|$ . (In symmetric notation,  $\langle x, T^*y^* \rangle := \langle Tx, y^* \rangle := y^*(Tx)$ .)

## Dual space of $C[a, b]$

Let  $[a, b]$  be a closed bounded interval in  $\mathbb{R}$ . Let  $C[a, b]$  be the space of  $\mathbb{R}$ -valued continuous functions on  $[a, b]$  with the sup-norm  $\|\cdot\|_\infty$ .

Let  $\rho: [a, b] \rightarrow \mathbb{R}$  be a real-valued function and  $P: \{a = x_0 < \dots < x_n = b\}$  be a partition of  $[a, b]$ . Define the variation of  $\rho$  with respect to  $P$  by

$$V(\rho, P) := \sum_{k=1}^n |\rho(x_k) - \rho(x_{k+1})|,$$

and the *total variation* by

$$V(\rho) := \sup_{P \in \mathcal{P}} V(\rho, P)$$

where  $\mathcal{P}$  denotes all the partitions of  $[a, b]$ . A function  $\rho: [a, b] \rightarrow \mathbb{R}$  is called *bounded variation* if  $V(\rho) < \infty$ . Let  $BV[a, b]$  denote the vector space of all the bounded variations.

Let  $f \in C[a, b]$  and  $\rho \in BV[a, b]$ . Let  $P : a = x_0 < \dots < x_n = b$  with tags  $t_k \in [x_{k-1}, x_k]$  be a tagged partition. Define the *Riemann-Stieltjes sum* with respect to  $\rho$  and  $P$  by

$$S(f, \rho, P) = \sum_{k=1}^n f(t_k) \left( \rho(x_k) - \rho(x_{k-1}) \right).$$

Then the *Riemann-Stieltjes integral* is defined by

$$\int_a^b f(x) d\rho(x) := \lim_{\|P\| \rightarrow 0} S(f, \rho, P).$$

where  $\|P\|$  denotes the diameter of a partition. The Riemann-Stieltjes integral exists by the uniform continuity of  $f$  on  $[a, b]$ .

Observe that  $V(\cdot)$  satisfies non-negativity, scaling property and the triangle inequality. However,  $V(\cdot)$  is not non-degenerate since  $V(\rho) = 0$  only implies that  $\rho$  is constant on  $[a, b]$ . Hence we restrict to the following subspace (the notation may not be standard)

$$BV_0[a, b] = \{ \rho \in BV[a, b] : \rho(a) = 0 \}.$$

Then it is readily checked that  $BV_0[a, b]$  is a Banach space under the norm  $V(\cdot)$ .

To justify the injectivity in our proof, we further remove the redundancy and modify the space to

$$BV_0^+[a, b] := \left\{ \rho \in BV_0[a, b] : \lim_{y \rightarrow x^+} \rho(y) = \rho(x), \forall x \in (a, b) \right\}. \quad (1)$$

It can be checked that  $BV_0^+[a, b]$  is closed in  $BV_0[0, 1]$  since the right continuity is preserved by uniform convergence and  $\|\cdot\|_\infty \leq V(\cdot)$  on  $BV_0[a, b]$ .

*Remark.* The elements in (1) are defined explicitly. They can be viewed as representatives of classes in a quotient space whose definition shares the same purpose to establish the injectivity. The details are given in the next section.

It follows from Jordan decomposition of  $\rho \in BV[a, b]$  that  $\rho = \rho_+ - \rho_-$  where  $\rho_+$  and  $\rho_-$  are increasing. Hence for  $x \in (a, b)$ ,

$$\rho^* := \lim_{y \rightarrow x^+} \rho = \lim_{y \rightarrow x^+} \rho_+(x) - \lim_{y \rightarrow x^+} \rho_-(x) \quad (2)$$

is well defined, i.e.,  $\rho^* \in BV_0^+[a, b]$ . Since  $\rho_+$  and  $\rho_-$  are monotone,  $\rho^*$  and  $\rho$  only differ on the at most countable discontinuities in  $(a, b)$ . Moreover, for  $\rho \in BV_0[a, b]$ ,

$$V(\rho^*) \leq V(\rho) \text{ and } \int_a^b f d\rho = \int_a^b f d\rho^* \text{ for all } f \in C[a, b] \quad (3)$$

and for  $\rho_1, \rho_2 \in BV_0^+[a, b]$ ,

$$\text{if } \int_a^b f d\rho_1 = \int_a^b f d\rho_2 \text{ for all } f \in C[a, b], \text{ then } \rho_1 = \rho_2 \in BV_0^+[a, b]. \quad (4)$$

The proofs of (3) and (4) are given in Appendix. After the above modifications, we are ready to state the main theorem.

**Theorem 1.** Under above notation,  $C[a, b]^* = BV_0^+[a, b]$ .

*Proof.* We first introduce some convenient notation. For  $f \in C[a, b]$  and  $\rho \in BV_0[a, b]$ , denote

$$\langle f, \rho \rangle := \int_a^b f d\rho.$$

Then  $\langle \cdot, \cdot \rangle: C[a, b] \times BV_0[a, b] \rightarrow \mathbb{R}$  is well defined by the existence of Riemann-Stieltjes integral. It follows from the linearity of summation that for  $\alpha \in \mathbb{R}$ ,  $f, \tilde{f} \in C[a, b]$  and  $\rho, \tilde{\rho} \in BV_0[a, b]$ ,

$$\langle \alpha f + \tilde{f}, \rho \rangle = \alpha \langle f, \rho \rangle + \langle \tilde{f}, \rho \rangle \text{ and } \langle f, \alpha \rho + \tilde{\rho} \rangle = \alpha \langle f, \rho \rangle + \langle f, \tilde{\rho} \rangle. \quad (5)$$

And since for any partition  $P$  with tags, we have

$$\left| \sum_{k=1}^n f(t_k) (\rho(x_k) - \rho(x_{k-1})) \right| \leq \|f\|_\infty \sum_{k=1}^n |\rho(x_k) - \rho(x_{k-1})| = \|f\|_\infty V(\rho, P) \leq \|f\|_\infty V(\rho).$$

Then take the limit  $\|P\| \rightarrow 0$  on the LHS, we have

$$|\langle f, \rho \rangle| \leq \|f\|_\infty V(\rho). \quad (6)$$

By (5) and (6), for any fixed  $\rho \in BV_0^+[a, b]$ , the map  $\langle \cdot, \rho \rangle: C[a, b] \rightarrow \mathbb{R}$  is linear and bounded, i.e.,  $\langle \cdot, \rho \rangle \in C[a, b]^*$ . To complete the proof, we will prove the map

$$\begin{aligned} T: BV_0^+[a, b] &\rightarrow C[a, b]^* \\ \rho &\mapsto \langle \cdot, \rho \rangle \end{aligned}$$

is an isometric isomorphism.

- (i) (linear and injective) By (5),  $T$  is linear. By (4),  $T$  is injective.
- (ii) (surjective) For any  $\Lambda \in C[a, b]^*$ , we will first find  $\rho \in BV_0[a, b]$  such that  $\Lambda f = \langle f, \rho \rangle$  for all  $f \in C[a, b]$  and then modify  $\rho$  to  $\rho^* \in BV_0^+[a, b]$ .

(Inspired by the ‘formal’ argument that  $\rho(x) - \rho(a) = \int_{[a,x]} d\rho = \langle \chi_{[a,x]}, \rho \rangle \approx \Lambda \chi_{[a,x]}$ . But we can NOT apply  $\Lambda$  directly to  $\chi_{[a,x]}$  since  $\chi_{[a,x]}$  is not continuous, which is where *Hahn-Banach* comes into the stage.) Observing that  $C[a, b]$  is a subspace in the normed space  $B[a, b]$  of bounded functions, we apply *Hahn-Banach theorem* to extend  $\Lambda$  to  $\tilde{\Lambda} \in B[a, b]^*$  with  $\|\tilde{\Lambda}\| = \|\Lambda\|$ . Hence we are able to define  $\rho(x) := \tilde{\Lambda} \chi_{[a,x]}$  for  $x \in (a, b]$  and  $\rho(0) := 0$ .

First we check  $\rho \in BV_0[a, b]$ . For any partition  $P$ , write  $\theta_k = \text{Sgn}(\rho(x_k) - \rho(x_{k-1}))$ . Then by the linearity and  $\|\tilde{\Lambda}\| = \|\Lambda\|$ ,

$$\begin{aligned} \sum_{k=1}^n |\rho(x_k) - \rho(x_{k-1})| &= \sum_{k=1}^n \theta_k (\rho(x_k) - \rho(x_{k-1})) \\ &= \theta_1 \tilde{\Lambda} \chi_{[a,x_1]} + \sum_{k=2}^n \theta_k (\tilde{\Lambda} \chi_{[a,x_k]} - \tilde{\Lambda} \chi_{[a,x_{k-1}]}) \\ &= \tilde{\Lambda} \left( \theta_1 \chi_{[a,x_1]} + \sum_{k=2}^n \theta_k (\chi_{[a,x_k]} - \chi_{[a,x_{k-1}]}) \right) \\ &\leq \|\tilde{\Lambda}\| \left\| \theta_1 \chi_{[a,x_1]} + \sum_{k=2}^n \theta_k \chi_{[x_{k-1}, x_k]} \right\|_\infty \\ &= \|\Lambda\|, \end{aligned} \quad (7)$$

where in the last equality holds since the function is of sup-norm 1. Take supremum over partition  $P$  on LHS to obtain  $V(\rho) \leq \|\Lambda\|$ . Hence  $\rho \in BV_0[a, b]$ .

Next we check  $\Lambda f = \langle f, \rho \rangle$  for all  $f \in C[a, b]$ .

(Inspired by the facts that Riemann-Stieljes integral is continuous w.r.t.  $\|\cdot\|_\infty$  and  $f$  can be uniformly approximated by step functions.) Let  $\varepsilon > 0$ . By the uniform continuity of  $f$ , there exists  $\delta_1$  such that for any partition  $P$  with  $\|P\| \leq \delta_1$ ,  $\sup_{x \in [x_{k-1}, x_k]} |f(x) - f(x_k)| \leq \varepsilon$  for  $1 \leq k \leq n$ . Then define the step function  $\tilde{f} = f(x_1)\chi_{[a, x_1]} + \sum_{k=2}^n f(x_k)\chi_{(x_{k-1}, x_k]}$

$$\|f - \tilde{f}\|_\infty \leq \varepsilon \tag{8}$$

By the definition of Riemann-Stieljes integral, there exists  $\delta_2 > 0$ , such that for any partition  $P$  with  $\|P\| \leq \delta_2$  and the tags  $t_k = x_k, 1 \leq k \leq n$ , we have

$$|\langle f, \rho \rangle - S(f, \rho, P)| \leq \varepsilon. \tag{9}$$

It follows from a similar check in (7) that  $S(f, \rho, P) = \tilde{\Lambda}\tilde{f}$ . Hence when  $\|P\| < \min\{\delta_1, \delta_2\}$ , by (8) and (9),

$$\begin{aligned} |\langle f, \rho \rangle - \Lambda f| &= \left| \langle f, \rho \rangle - \tilde{\Lambda}\tilde{f} \right| \\ &\leq \left| \langle f, \rho \rangle - \tilde{\Lambda}\tilde{f} \right| + |\tilde{\Lambda}\tilde{f} - \tilde{\Lambda}f| \\ &\leq |\langle f, \rho \rangle - S(f, \rho, P)| + \|\tilde{\Lambda}\| \|f - \tilde{f}\|_\infty \\ &\leq (1 + \|\Lambda\|)\varepsilon. \end{aligned}$$

Letting  $\varepsilon \rightarrow 0$ , we have  $\langle f, \rho \rangle = \Lambda f$ .

Replace  $\rho$  with  $\rho^*$  defined in (2). It follows from (3) that  $\langle f, \rho^* \rangle = \langle f, \rho \rangle = \Lambda f$  and  $V(\rho^*) \leq V(\rho) \leq \|\Lambda\|$ .

- (iii) (isometric) Let  $\rho \in BV_0^+[a, b]$ . It follows from (6), that  $\|T\rho\| \leq V(\rho)$ . By (ii), there exists  $\tilde{\rho} \in BV_0^+[a, b]$  with  $V(\tilde{\rho}) \leq \|T\rho\|$  and  $\langle f, \tilde{\rho} \rangle = \langle f, \rho \rangle$ . It follows from (i) that  $\rho = \tilde{\rho}$ . Hence  $V(\rho) = V(\tilde{\rho}) \leq \|T\rho\|$ , thus  $V(\rho) = \|T\rho\|$ .

□

*Remark.* A similar proof shows [Theorem 1](#) also holds when the scalar field is  $\mathbb{C}$ . These results are the special cases of Riesz representation of  $C_0(X)^*$  via Borel regular measures when  $X$  is locally compact Hausdorff.

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It's good to stop here.

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*Remark.* Actually the explicit candidate in (1) is found in a ‘cheated’ way. We can reason as following, for  $\Lambda \in C[a, b]^*$ , by the general Riesz representation, there exists a unique Borel regular measure  $\mu \in M[a, b]$  such that  $\Lambda f = \int_{[a, b]} f d\mu$  (the integral is defined in Lebesgue way). Then the cumulative distribution function  $F_\mu(t) := \mu[a, t]$  is right continuous (by the continuity of measure) and of bounded variation. Moreover,  $\mu$  is the measure extension of the premeasure induced by  $F_\mu$  on semiring  $\{a, \emptyset, (c, d], a \leq c < d \leq b\}$ . However, notice that for the Dirac measure  $\delta_a$  (representing the evaluation  $\Lambda f = f(a)$ ), the Riemann-Stieljes integral w.r.t.  $F_{\delta_a} = \chi_{[a, b]}$

identically vanish on  $C[a, b]$  ! Then we realize the Riemann-Stieltjes integral will ‘forget’ the jump at  $a$  if the  $\rho$  is right-continuous at  $a$ . Hence we made the following modification for  $\mu \in M[a, b]$ ,

$$\Lambda f = \int_{[a,b]} f d\mu = \mu\{a\}f(a) + \int_a^b f dF_\mu = \int_a^b f d(F_\mu + G_{\mu\{a\}})$$

where  $G_{\mu\{a\}}(a) := -\mu\{a\}\chi_{\{a\}}$ . Hence  $\widetilde{F}_\mu := F_\mu + G_{\mu\{a\}} \in BV_0^+[a, b]$ .

## A quotient space perspective

Instead of explicitly finding the representatives like (1), another natural way to achieve the injectivity is to define a quotient space. First, we define a subspace of  $BV_0[a, b]$  as

$$H := \{\rho \in BV_0[a, b] : \langle f, \rho \rangle = 0, \forall f \in C[a, b]\}. \tag{10}$$

By (6),  $H$  is closed as the intersection of the kernels of continuous function  $\langle f, \cdot \rangle$ . Explicitly,  $H$  is exactly the subspace of  $BV_0[a, b]$  consisting of the functions differing from 0 only on at most countable points on  $(a, b)$ . Hence  $BV_0[a, b]/H$  is well-defined. Let  $\pi$  be the natural projection. Define  $T: BV_0[a, b]/H \rightarrow C[a, b]^*$  by  $T(\pi(\rho)) = \langle \cdot, \rho \rangle$ . Recall that  $\|\pi(\rho)\| \leq \|\rho\|$  for  $\pi(\rho) \in BV_0[a, b]/H$  and  $|\langle f, \rho \rangle| = |\langle f, \rho + h \rangle| \leq \|f\|_\infty \|\rho + h\|$  for all  $h \in H$ . Then taking infimum with respect to  $h \in H$  leads to  $\|T\pi(\rho)\| \leq \|\pi(\rho)\|$ . The linear and injectivity follows as we expected. The surjectivity is obtained in the same way in the proof of [Theorem 1](#). Thus

$$C[a, b]^* = BV_0[a, b]/H.$$

## Appendix

In this Appendix, we will establish the intuition that for  $f \in C[a, b]$ , the Riemann-Stieltjes integral  $\langle f, \rho \rangle$  is invariant under a change of values at the countable **interior** discontinuities of  $\rho \in BV_0[a, b]$ .

**Lemma 2.** *Let  $c \in (a, b)$  and  $\alpha \in \mathbb{K}$ . The Riemann-Stieltjes integral  $\int_a^b f d(\alpha\chi_{\{c\}}) = 0$  for all  $f \in C[a, b]$ .*

*Proof.* Let  $f \in C[a, b]$  and  $P$  be any tagged partition of  $[a, b]$ . If  $c$  is in the interior of some  $[x_k, x_{k-1}]$ , then  $S(f, \chi_{\{c\}}, P) = 0$ . If  $c = x_k$  for some  $x_k \in (a, b)$ , then by choosing the tag  $c = x_k$  at both  $[x_{k-1}, x_k]$  and  $[x_k, x_{k+1}]$ , we have  $S(f, \chi_{\{c\}}, P) = \alpha f(c) - \alpha f(c) = 0$ . Hence  $\int_a^b f d(\alpha\chi_{\{c\}}) = 0$ . □

**Lemma 3.** *Let  $\rho \in BV_0[a, b]$ . Denote  $(c_n)_{n=1}^\infty$  the discontinuous points of  $\rho$  in  $(a, b)$  and  $(\alpha_n)_{n=1}^\infty$  the oscillations of  $\rho$ , more precisely,  $\alpha_n = |\lim_{y \rightarrow c_n^-} \rho(y) - \lim_{y \rightarrow c_n^+} \rho(y)|$ . For any sequence  $(\beta_n)_{n=1}^\infty$  such that  $|\beta_n| \leq \alpha_n$  for all  $n \in \mathbb{N}$ , define  $\eta = \sum_{n=1}^\infty \beta_n \chi_{\{c_n\}}$ . Then  $\eta \in BV_0[a, b]$  and  $\int_a^b f d\eta = 0$  for all  $f \in C[a, b]$ .*

*If  $\rho$  has only finitely many discontinuities, [Lemma 2](#) finishes the proof.*

*Proof.* Since  $\rho \in BV_0[a, b]$ ,  $\sum_{n=1}^{\infty} |\beta_n| \leq \sum_{n=1}^{\infty} \alpha_n \leq V(\rho) < \infty$ . Hence for any  $\varepsilon > 0$ , there exists  $N \in \mathbb{N}$  such that  $\sum_{n=N+1}^{\infty} |\beta_n| \leq \varepsilon/2$ . Then

$$V\left(\sum_{n=N+1}^{\infty} \beta_n \chi_{\{c_n\}}\right) \leq 2 \sum_{n=N+1}^{\infty} |\beta_n| \leq \varepsilon.$$

Let  $f \in C[a, b]$ . By [Lemma 2](#) and (6),

$$|\langle f, \eta \rangle| = \left| \left\langle f, \sum_{n=1}^N \beta_n \chi_{\{c_n\}} \right\rangle + \left\langle f, \sum_{n=N+1}^{\infty} \beta_n \chi_{\{c_n\}} \right\rangle \right| \leq 0 + \|f\|_{\infty} \varepsilon.$$

Letting  $\varepsilon \rightarrow 0$ , we have  $\langle f, \eta \rangle = 0$ . □

*Proof of (3).* Let  $\rho \in BV_0[a, b]$ . Define  $\rho_n := \begin{cases} \rho(x + 1/n) & \text{if } x \in [a, b - 1/n] \\ \rho(b) & \text{if } x \in (b - 1/n, b]. \end{cases}$  Then it is readily checked that  $V(\rho_n) \leq V(\rho)$  and  $\rho^* = \lim_{n \rightarrow \infty} \rho_n$ . By the lower semi-continuity of  $V(\cdot)$  (see e.g. [Royden-Fitzpatrick Real Analysis, Sec 6.3 Problem 33]),  $V(\rho^*) \leq V(\rho)$ .

By the definition of  $\rho^*$ , we have  $\rho^* - \rho = \sum_{n=1}^{\infty} \beta_n \chi_{\{c_n\}}$  for some sequence  $(\beta_n)_{n=1}^{\infty}$  satisfying the condition in [Lemma 3](#). Hence  $\langle f, \rho^* \rangle = \langle f, \rho \rangle$  for all  $f \in C[a, b]$ . □

*Proof of (4).* It suffices to prove that if  $\rho \in BV_0^+[a, b]$  and  $\langle f, \rho \rangle = \int_a^b f d\rho = 0$  for all  $f \in C[a, b]$ , then  $\rho = 0$ . Define a right continuous function  $\rho^*$  by  $\rho^*(a) := \lim_{x \rightarrow a^+} \rho(x)$  and  $\rho^*(x) = \rho(x)$  for  $x \in (a, b]$ . Then we can generate a measure  $\mu$  on  $[a, b]$  by setting  $\mu[a, x] = \rho^*(x)$ .

For  $n \in \mathbb{N}$ , let  $f_n(x) := \begin{cases} 1 & [a, c] \\ \text{linear} & (c, c + 1/n] \\ 0 & (c + 1/n, b] \end{cases}$ . The definition of Riemann-Stieljes integral

implies that  $\int_a^b f_n d(\rho^*(a) \chi_{\{a\}}) = -\rho^*(a) f_n(a)$  and recall  $\rho^* = \rho + \rho^*(a) \chi_{\{a\}}$ . Then for any  $c \in (a, b]$ ,

$$\begin{aligned} \rho^*(c) = \mu[a, c] &= \lim_{n \rightarrow \infty} \int f_n d\mu = \lim_{n \rightarrow \infty} \left( \rho^*(a) f_n(a) + \int_a^b f_n d\rho^* \right) \\ &= \lim_{n \rightarrow \infty} \left( \rho^*(a) f_n(a) + \int_a^b f_n d\rho + \int_a^b f_n d(\rho^*(a) \chi_{\{a\}}) \right) = 0 \end{aligned}$$

where the second equality is by Lebesgue dominated convergence theorem. Hence  $\rho^* = 0$  on  $(a, b]$ , and so  $\rho = 0$  on  $[a, b]$ . □