THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH4010 Functional Analysis 2022-23 Term 1

Solution to Homework 7

1. Let (x_n) be a sequence in an inner product space. Show that the conditions $\|x_n\| \to \|x\|$ and $\langle x_n, x\rangle \to \langle x, x\rangle$ imply $x_n \to x$.

   Proof. Note that

   $$\|x - x_n\|^2 = \langle x - x_n, x - x_n\rangle = \|x\|^2 - 2\Re\langle x_n, x\rangle + \|x_n\|^2.$$

   By $\langle x_n, x\rangle \to \langle x, x\rangle$, we have $\Re\langle x_n, x\rangle \to \Re\langle x, x\rangle = \langle x, x\rangle$ since the real part map $\Re\colon \mathbb{C} \to \mathbb{R}$ is continuous. Together with $\|x_n\| \to \|x\|$,

   $$\|x - x_n\|^2 \to \|x\|^2 - 2\langle x, x\rangle + \|x\|^2 = 0, \quad \text{as } n \to \infty.$$

   Thus $x_n \xrightarrow{\|\cdot\|} x$. □

2. Prove that in a complex (resp. real) inner product space, $x \perp y$ if and only if

   $$\|x + \lambda y\| = \|x - \lambda y\| \tag{1}$$

   for all scalars $\lambda \in \mathbb{C}$ (resp. $\mathbb{R}$).

   Proof. Let X denote an inner product space with scalar field $\mathbb{K}$, where $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$.

   ($\Longrightarrow$) Let $x, y \in X$. If $x \perp y$, then $x \perp \pm\lambda y$ for all $\lambda \in \mathbb{K}$. By Pythagorean theorem,

   $$\|x + \lambda y\|^2 = \|x\|^2 + \|\lambda y\|^2 = \|x - \lambda y\|^2.$$

   ($\Longleftarrow$) By Polarization identities, for $x, y \in X$, if $\mathbb{K} = \mathbb{R}$, then

   $$\langle x, y\rangle = \frac{1}{4}(\|x + y\|^2 - \|x - y\|^2), \tag{2}$$

   and if $\mathbb{K} = \mathbb{C}$, then

   $$\langle x, y\rangle = \frac{1}{4}\left(\|x + y\|^2 - \|x - y\|^2 + i\|x + iy\|^2 - i\|x - iy\|^2\right). \tag{3}$$

   Hence if $\mathbb{K} = \mathbb{R}$, then by taking $\lambda = 1$ in (1), it follows from (2) that $\langle x, y\rangle = 0$; If $\mathbb{K} = \mathbb{C}$, then by taking $\lambda = 1$ and i in (1), it follows from (3) that $\langle x, y\rangle = 0$. □

3. (a) Prove that for every two subspaces X_1 and X_2 of a Hilbert space,

   $$(X_1 + X_2)^\perp = X_1^\perp \cap X_2^\perp.$$

   (b) Prove that for every two closed subspaces X_1 and X_2 of a Hilbert space,

   $$(X_1 \cap X_2)^\perp = \overline{X_1^\perp + X_2^\perp}.$$

Proof. (a) It follows from $X_1, X_2 \subset (X_1 + X_2)$ that $(X_1 + X_2)^\perp \subset (X_1)^\perp, (X_2)^\perp$. Hence $(X_1 + X_2)^\perp \subset (X_1)^\perp \cap (X_2)^\perp$.

On the other hand, let $x^* \in (X_1)^\perp \cap (X_2)^\perp$. Then for $y \in X_1 + X_2$ with $y = x_1 + x_2$ for some $x_1 \in X_1, x_2 \in X_2$,

$$\langle y, x^* \rangle = \langle x_1 + x_2, x^* \rangle = \langle x_1, x^* \rangle + \langle x_2, x^* \rangle = 0.$$

This shows $x^* \in (X_1 + X_2)^\perp$, thus $(X_1)^\perp \cap (X_2)^\perp \subset (X_1 + X_2)^\perp$.

(b) Since X_1, X_2 are closed, we have $(X_i^\perp)^\perp = X_i$ for $i = 1, 2$. Applying (a) to $X_1^\perp$ and $X_2^\perp$ gives

$$(X_1^\perp + X_2^\perp)^\perp = (X_1^\perp)^\perp \cap (X_2^\perp)^\perp = X_1 \cap X_2.$$

Hence

$$\overline{X_1^\perp + X_2^\perp} = \big((X_1^\perp + X_2^\perp)^\perp\big)^\perp = (X_1 \cap X_2)^\perp.$$

□

— *THE END* —