THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH4010 Functional Analysis 2022-23 Term 1

Solution to Homework 4

1. Prove that for every x in a normed space X, the following identity holds:

$$\|x\| = \sup\left\{\frac{|f(x)|}{\|f\|} : f \in X^*,\ f \neq 0\right\}.$$

Proof. Let $x \in X$. Since $|f(x)| \leq \|f\|\|x\|$ by the definition of $\|f\|$, we have

$$\frac{|f(x)|}{\|f\|} \leq \|x\| \quad \text{for } f \in X^* \setminus \{0\}.$$

Hence LHS $\geq$ RHS. On the other hand, by Hahn-Banach theorem there exists $f \in X^*$ with $\|f\| = 1$ such that

$$\|x\| = |f(x)|.$$

This finishes the proof. □

2. Let $C[0,1]$ be the vector space of continuous functions on $[0,1]$. Define $\delta(x) = x(0)$ for $x \in C[0,1]$.

   (a) Show that δ is a bounded linear functional if $C[0,1]$ is endowed with the sup-norm. Find the norm of δ.

   (b) Show that δ is an unbounded linear functional if $C[0,1]$ is endowed with the norm

$$\|x\|_1 = \int_0^1 |x(t)|\,dt. \tag{1}$$

Proof. Let $\|\delta\|_\infty$ and $\|\delta\|_1$ denote the norms of δ with $C[0,1]$ endowed with norms $\|\cdot\|_\infty$ and $\|\cdot\|_1$ respectively, where $\|\cdot\|_1$ is defined in (1). The linearity of δ is easy to verify.

(a) By the definition of δ,

$$|\delta(x)| = |x(0)| \leq \|x\|_\infty.$$

Then $\|\delta\|_\infty \leq 1$. On the other hand, let $x_0 = 1$ on $[0,1]$, then $|\delta(x_0)| = 1 = \|x_0\|_\infty$. Hence $\|\delta\|_\infty = 1$.

(b) For $n \in \mathbb{N}_{\geq 2}$, define

$$x_n(t) := \begin{cases} -\frac{n^2}{2}t + n & \text{if } t \in [0, \frac{2}{n}] \\ 0 & \text{if } t \in (\frac{2}{n}, 1]. \end{cases}$$

Then $\|x_n\|_1 = 1$ but $|\delta(x_n)| = |x_n(0)| = n$. Hence $\|\delta\|_1 \geq n$. Letting $n \to \infty$ shows that δ is unbounded.

□

— *THE END* —