Math 2050, highlight of Week 1

1. Motivating Question:

What is the system of real numbers $\mathbb{R}$?

1.1. **Algebraic properties of Real numbers.** We start with the algebraic properties of $(\mathbb{R}, +, \cdot)$:

(a1) $\forall a, b \in \mathbb{R}$, we have $a + b = b + a$;
(a2) $\forall a, b, c \in \mathbb{R}$, we have $(a + b) + c = a + (b + c)$;
(a3) $\exists 0 \in \mathbb{R}$ such that $a + 0 = 0 + a = a$ for all $a \in \mathbb{R}$;
(a4) $\forall a \in \mathbb{R}$, there exists $b \in \mathbb{R}$ such that $a + b = b + a = 0$.
(m1) $\forall a, b \in \mathbb{R}$, we have $a \cdot b = b \cdot a$;
(m2) $\forall a, b, c \in \mathbb{R}$, we have $(a \cdot b) \cdot c = a \cdot (b \cdot c)$;
(m3) $\exists 1 \neq 0 \in \mathbb{R}$ such that $a \cdot 1 = 1 \cdot a = a$ for all $a \in \mathbb{R}$;
(m4) $\forall a \neq 0 \in \mathbb{R}$, there exists $b \in \mathbb{R}$ such that $a \cdot b = b \cdot a = 1$.
(d) For all $a, b, c \in \mathbb{R}$, we have $a \cdot (b + c) = a \cdot b + a \cdot c$.

Theorem 1.1. *(uniqueness) From the algebraic properties of $\mathbb{R}$, we have the following uniqueness of elements:*

(i) *If $a, b \in \mathbb{R}$ are elements such that $a + b = a$, then $b = 0$.*
(ii) *If $a, b \in \mathbb{R}$ are elements such that $a \neq 0$ and $a \cdot b = a$, then $b = 1$.*
(iii) *Given $a \in \mathbb{R}$. If $b, c \in \mathbb{R}$ are such that $a + b = 1 = a + c$, then $b = c$.*
(iv) *Given $0 \neq a \in \mathbb{R}$. If $b, c \in \mathbb{R}$ are such that $a \cdot b = 1 = a \cdot c$, then $b = c$.*

Remark 1.1. The importance of this Theorem is that the "zero" and "identity" elements are unique. Moreover, the additive and multiplicative inverse are unique. And hence, we may use $-a$ and a^{-1} to denote the inverse respectively.

With the inverse defined, we may proceed to define the "negative" operation. Namely, the subtraction:

$$a - b = a + (-b), \quad \forall a, b \in \mathbb{R};$$

and division:

$$a/b = a \cdot (b^{-1}), \quad \forall a, b \in \mathbb{R}, \ b \neq 0.$$

1.2. **Ordering properties of Real numbers.** Next, we would like to define a ordering properties of the real number which enables us to compare elements as well as define the inequalities, distance, etc. To do this, we let $\mathbb{P}$ be a subset of $\mathbb{R}$ such that the following holds:

(i) If $a \in \mathbb{R}$, then either $a = 0$, $a \in \mathbb{P}$ or $-a \in \mathbb{P}$;
(ii) if $a, b \in \mathbb{P}$, then $a + b \in \mathbb{P}$;
(iii) if $a, b \in \mathbb{P}$, then $a \cdot b \in \mathbb{P}$.

We call the set $\mathbb{P}$ to be set of positive numbers. With this definition, all simple inequality will hold (Check!).

With this in hand, we also define distance between $x, y \in \mathbb{R}$ by $|x-y|$ where $|a|$ of a real number $a \in \mathbb{R}$ is given by

$$|a| = \begin{cases} a, & \text{if } a \geq 0; \\ -a, & \text{if } a < 0. \end{cases} \tag{1.1}$$

1.3. **Distinction between $\mathbb{R}$ and $\mathbb{Q}$.** In view of algebraic properties, we can see the necessity of improving the number systems:

- Natural number $\mathbb{N}$: Fail to obey addition rule;
- Integers $\mathbb{Z}$: Fail to obey multiplicative rule;
- Rational number $\mathbb{Q}$: Satisfies all rule!

Problem raised:

Theorem 1.2. *There is no $x \in \mathbb{Q}$ such that $x^2 = 2$.*

1.4. **Completeness of $\mathbb{R}$.**

Definition 1.1. *Let $A \subset \mathbb{R}$ be a subset, we say that A is bounded from above if there is $M \in \mathbb{R}$ such that for all $a \in A$, $a \leq M$.*

Analogously, we can define the notion of "bounded below" and "bounded". Clearly, there are no unique upper bound for a bounded set. We therefore look for the "best" one.

Definition 1.2. *Given a non-empty subset $S \subset \mathbb{R}$ which is bounded from above. A real number $u = \sup S$ (the least upper bound) if*

(1) *u is an upper bound of S;*
(2) *If v is another upper bound of S, then $v \geq u$.*

Remark: by (2), $\sup S$ is unique if exists.

The greatest lower bound ($\inf S$) is defined analogously for non-empty subset S which is bounded from below.

The completeness of $\mathbb{R}$:

For any non-empty subset S which is bounded from above, $\sup S$ *exists.*

Corollary 1.1. *For any non-empty subset S which is bounded from below,* $\inf S$ *exists.*

Important applications of completeness:

Theorem 1.3 (Archimedean properties)**.** *The set of natural number $\mathbb{N}$ is unbounded.*

Another crucial consequence to our motivating question (!!!):

Theorem 1.4. *There is $u \in \mathbb{R}$ such that $u^2 = 2$.*

Proof. Let $A = \{a \in \mathbb{R} : a^2 < 2\}$. The set A is non-empty since $0^2 = 0 < 2$ and hence $0 \in A$. Moreover, A is bounded from above by 2 since otherwise, there is $a > 2$ such that $4 < a^2 < 2$ which is impossible.

Therefore, completeness implies that $u = \sup A$ exists in $\mathbb{R}$. Since $1^2 = 1 < 2$, we have $u \geq 1 \in A$. We claim that $u^2 = 2$. Suppose on the contrary, we either have $u^2 < 2$ or $u^2 > 2$.

Case 1. $u^2 > 2$. We choose $\varepsilon > 0$ to be

$$\varepsilon = \min\left\{\frac{u^2-2}{2u}, u\right\} > 0.$$

By properties of $\sup A$, there is $a \in A$ such that $0 < u - \varepsilon < a$ and hence

$$(u-\varepsilon)^2 < a^2 < 2.$$

But the number $v = u - \varepsilon$ satisfies

$$v^2 = u^2 - 2\varepsilon u + \varepsilon^2 > u^2 - 2\varepsilon u > 2$$

which is impossible.

Case 2. $u^2 < 2$. We choose $\varepsilon > 0$ to be

$$\varepsilon = \min\left\{1, \frac{2-u^2}{2(2u+1)}\right\} > 0$$

Then the number $v = u + \varepsilon$ satisfies

$$\begin{aligned} v^2 &= u^2 + 2\varepsilon u + \varepsilon^2 \\ &\leq u^2 + \varepsilon(2u+1) \\ &\leq u^2 + \frac{2-u^2}{2} \\ &< 2. \end{aligned}$$

Therefore, $v \in A$ and hence $u + \varepsilon \leq u$ which is impossible. $\square$

Math 2050, quick note of Week 2

1. Density of Rational and Irrational numbers on $\mathbb{R}$

From numerical point of view, we approximate $\sqrt{2}$ by 1.41421356237....Precisely, what we are doing is: finding a sequence of rational number, namely

$$\begin{cases} a_1 = 1; \\ a_2 = 1.4; \\ a_3 = 1.41; \\ a_4 = 1.414... \end{cases} \tag{1.1}$$

so that a_n gets closer and closer to "THE" number $\sqrt{2}$ which is the abstract number obtained from completeness. This suggests a density nature of $\mathbb{Q}$. And here is the general result.

Theorem 1.1 (Density of rational number)**.** *For all $x, y \in \mathbb{R}$ such that $x < y$, we can find $q \in \mathbb{Q}$ such that $q \in (x, y)$.*

Example: We have

$$\sup\{q \in \mathbb{Q} : q^2 < 2, q > 0\} = \sqrt{2} \in \mathbb{R} \setminus \mathbb{Q}.$$

(We can think of $\mathbb{R}$ as the minimal completion of $\mathbb{Q}$ so that the "missing hole" is filled.)

And similarly, the irrational number is also dense.

Theorem 1.2 (Density of rational number)**.** *For all $x, y \in \mathbb{R}$ such that $x < y$, we can find $q \notin \mathbb{Q}$ such that $q \in (x, y)$.*

And hence irrational number are also "almost everywhere" inside $\mathbb{R}$.

2. Intervals

For notational convenience, we will use

(1) $(a, b) = \{x : a < x < b\}$;
(2) $[a, b) = \{x : a \le x < b\}$;
(3) $(a, b] = \{x : a < x \le b\}$;
(4) $[a, b] = \{x : a \le x \le b\}$;
(5) $(a, +\infty) = \{x : a < x\}$;
(6) $[a, +\infty) = \{x : a \le x\}$;
(7) $(-\infty, b) = \{x : x < b\}$;
(8) $(-\infty, b] = \{x : x \le b\}$;
(9) $(-\infty, +\infty) = \mathbb{R}$.

Hence, we can rephrase density as "Any non-empty open interval contains element in $\mathbb{Q}$ and $\mathbb{Q}^c$."

Question: How do we determine whether a subset of $\mathbb{R}$ is a interval or not?

Theorem 2.1 (Characterization of Interval)**.** *If S is a non-empty subset of $\mathbb{R}$ such that S contains two distinct real numbers and satisfies the following property:*

$$\text{For any } x, y \in S, \text{ we have } [x, y] \subset S;$$

then S is an interval.

2.1. **Special type of intervals.** For a sequence of interval $\{I_n\}_{n=1}^{\infty}$. We say that the sequence is nested if

$$I_k \subset I_{k-1}$$

for all $k \geq 1$. In particular, the sequence is "decreasing".

Example: $I_n = (0, \frac{1}{n})$, then $\cap_{n=1}^{\infty} I_n = \emptyset$. This is because if $x \in I_n$ for all n, then

$$0 < x < \frac{1}{n}.$$

But this contradicts with the Archimedean property.

Example: $I_n = [0, \frac{1}{n})$, then $\cap_{n=1}^{\infty} I_n = \{0\}$ since for $x \in \cap_{n=1}^{\infty} I_n$, we have for all n that

$$0 \leq x < \frac{1}{n}.$$

Clearly, 0 satisfies the above. And from Archimedean property, positive number fails to satisfies it and hence the assertion holds.

Example: $I_n = [n, +\infty)$, then $\cap_{n=1}^{\infty} I_n = \emptyset$ since for $x \in \cap_{n=1}^{\infty} I_n$, we have for all n that

$$x \geq n$$

which contradicts with the Archimedean property.

The above examples show that for a nested interval to have common intersection, it is necessary that

(a) I_n are bounded;
(b) I_n are closed,

for all n. It turns out to be sufficient as well:

Theorem 2.2 (Nested Interval Theorem)**.** *Suppose $\{I_n = [a_n, b_n]\}_{n=1}^{\infty}$ is a sequence of nested, closed and bounded interval on $\mathbb{R}$, then $\cap_{n=1}^{\infty} I_n$ is non-empty. Moreover, if $\inf\{b_n - a_n\} = 0$, then $\cap_{n=1}^{\infty} I_n$ is a singleton.*

Remark 2.1. For those who are interested in "Axiomatic" construction of $\mathbb{R}$, one can replace the completeness axiom of $\mathbb{R}$ by "Archimedean property and Nested Interval Property". The constructed $\tilde{\mathbb{R}}$ will be identical to the construction using completeness axiom. Google it if you want to know!

Theorem 2.3. $[0,1]$ *is uncountable.*

Proof. Suppose $[0,1]$ is countable. That is to say that the set $[0,1]$ is enumerative:

$$[0,1] = \{x_n\}_{n=1}^{\infty}.$$

Our goal is to construct some sequence which contradicts with something. We now construct a sequence of interval $\{I_n\}_{n=1}^{\infty}$ which are nested, closed and bounded.

Step 0. We choose $I_0 = [0,1]$.

Step 1. Considering $x_1 \in [0,1]$, we choose a subinterval $I_1 \subset I_0$ such that I_0 is closed and $x_1 \notin I_1$. This is possible since x_1 is simply a point!

Step 2. Considering $x_2 \in [0,1]$. If $x_2 \notin I_1$, then we take $I_2 = I_1$. Otherwise, we find a subinterval $I_2 \subset I_1$ such that I_2 is closed and $x_2 \notin I_2$.

....

Step $k, k > 2$**.** Consider $x_k \in [0,1]$. If $x_k \notin I_{k-1}$, then we take $I_k = I_{k-1}$. Otherwise, we find a subinterval $I_k \subset I_{k-1}$ such that I_k is closed and $x_k \notin I_k$.

(We are doing each steps ONE BY ONE!)

In this way, $\{I_n\}_{n=1}^{\infty}$ is a sequence of nested interval which are closed and bounded. Hence, Nested Interval Theorem implies $\eta \in \cap_{n=1}^{\infty} I_n \subset I_0 = [0,1]$. By our assumption, $\eta = x_N$ for some N since $[0,1] = \{x_n\}_{n=1}^{\infty}$. This implies

$$x_N \in I_N \cap I_N^c$$

which is impossible.

□

Math 2050, quick note of Week 3

1. Sequence and the convergence

We want to study the behaviour of sequence of real numbers, $\{a_n\}_{n=1}^{\infty}$. We want to study the concept of "limit" when $n \to +\infty$.

Definition 1.1. *Given a sequence of real number $\{a_n\}_{n=1}^{\infty}$.*

(i) *$\{a_n\}_{n=1}^{\infty}$ is said to be convergent to $a \in \mathbb{R}$ if $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that for all $n > N$, $|a - a_n| < \varepsilon$. In this case, we will write $\lim_{n\to+\infty} a_n = a$ or "$a_n \to a$ as $n \to +\infty$".*

(ii) *We say that $\{a_n\}_{n=1}^{\infty}$ is convergent if there is $a \in \mathbb{R}$ such that $\lim_{n\to+\infty} a_n = a$.*

(iii) *e say that $\{a_n\}_{n=1}^{\infty}$ is divergent if it is not convergent.*

Remark 1.1. In word (roughly speaking), the definition of convergent means that we can control the error ε as much as we wish as long as we consider sufficiently "late" element.

It is sometimes geometrically convenient to use

$$a_n \in V_\varepsilon(a) = \{x : |x - a| < \varepsilon\}$$

to emphasis that a_n is close to a with error at most ε.

To determine the convergence, it is only important to consider large index n. The following Sandwich Theorem illustrate this fact.

Theorem 1.1. *Suppose $\{a_n\}_{n=1}^{\infty}$ is a sequence such that $\lim_{n\to+\infty} a_n = 0$, $x, C \in \mathbb{R}, m \in \mathbb{N}$ and $\{x_n\}_{n=1}^{\infty}$ is a sequence such that for all $n > m$, we have*

$$|x - x_n| \le C a_n,$$

then we have $\lim_{n\to+\infty} x_n = x$.

We here give an example which used some common trick in analysis. (see more from the textbook)

Question 1.1. *Show that*

$$\lim_{n\to+\infty} \frac{n^2}{3^n} = 0.$$

Answer. Before we fix ε, let us do some estimate to simplify the question. For $n > 5$, we have

$$3^n = (1+2)^n \ge C_3^n 2^3 = n(n-1)(n-2) \cdot \frac{4}{3}. \tag{1.1}$$

Now, we used the fact that $n > 5$ to show that

$$\frac{n^2}{3^n} < \frac{n^2}{n(n-1)(n-2)} \le \frac{n^2}{n(n-\frac{n}{2})^2} = \frac{4}{n}.$$

Since $\lim_{n\to+\infty} 1/n = 0$ by Archimedean properties: For all $\varepsilon > 0$, there is N such that

$$\frac{1}{N} < \varepsilon.$$

And hence for all $n > N$, $n^{-1} < \varepsilon$. Now, we may apply Sandwich Theorem with $m = 5$, $C = 4$ and $a_n = 1/n$ to deduce the answer. $\square$

Using the above method, one can actually prove the following:

$$\lim_{n\to+\infty} \frac{P(n)}{(1+a)^n} = 0$$

for any polynomial $P(x)$ and $a > 0$ (Try it!).

We have some simply criterion for convergence.

Theorem 1.2. *Suppose $\{x_n\}_{n=1}^{\infty}$ is a convergent sequence, then $\{x_n\}_{n=1}^{\infty}$ is bounded.*

Important consequence: (!!!) Equivalently, if a sequence is unbounded, then the sequence is divergent! We will go back to this later.

The algebra operation is preserved under limiting process.

Theorem 1.3. *Suppose $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are two sequence of real number which $\lim_{n\to+\infty} x_n = x$ and $\lim_{n\to+\infty} y_n = y$. Then we have*

(1) $\lim_{n\to+\infty} x_n + y_n = x + y$;
(2) $\lim_{n\to+\infty} x_n - y_n = x - y$;
(3) $\lim_{n\to+\infty} x_n \cdot y_n = xy$;
(4) $\lim_{n\to+\infty} \frac{x_n}{y_n} = xy^{-1}$ *if* $y \neq 0$.

Math 2050, quick note of Week 4

1. Convergence and ordering

Preserving of ordering under convergence.

Theorem 1.1. *Suppose x_n and y_n are two sequence of real numbers such that $x_n \le y_n$ for all n. If $\lim_{n\to+\infty} x_n = x$ and $\lim_{n\to+\infty} y_n = y$, then $x \le y$.*

A simple consequence is the Squeeze theorem:

Theorem 1.2 (Squeeze theorem)**.** *Suppose x_n, y_n and z_n are sequences of real numbers such that*

$$x_n \le y_n \le z_n$$

for all $n \in \mathbb{N}$. If $\lim_{n\to+\infty} x_n = \lim_{n\to+\infty} z_n = L$, then $\{y_n\}$ is convergent with $\lim_{n\to+\infty} y_n = L$.

The upshot: The "closed" inequality will be preserved under convergence.
question: What about the opposite? Namely if the limit lies on some interval, is the tail of the sequence also lies inside it?

Theorem 1.3. *Suppose x_n is a sequence of real number such that $\lim_{n\to+\infty} x_n = x$. If $x \in (a, b)$ for some a, b, then there is $N \in \mathbb{N}$ such that for all $n > N$, $x_n \in (a, b)$.*

One of the application is the following special case:

Theorem 1.4. *Suppose x_n is a sequence of positive real number such that $\lim_{n\to+\infty} \frac{x_{n+1}}{x_n} < 1$, then $x_n \to 0$ as $n \to +\infty$.*

2. Criterion of convergence

We would like to determine the convergence of a particular sequence. By boundedness Theorem, a convergent sequence must be bounded.

Example: $x_n = (-1)^n$ is clearly bounded but divergent.

Question: What extra structure can guarantee the convergence?
We first consider a special type of sequences.

Definition 2.1.
1. *A sequence x_n is said to be increasing if $x_{n+1} \ge x_n$ for all n;*
2. *A sequence x_n is said to be decreasing if $x_{n+1} \le x_n$ for all n;*
3. *A sequence x_n is said to be monotone if it is either increasing or decreasing.*

In this case, the boundedness Theorem is also a sufficient condition.

Theorem 2.1 (Monotone convergence theorem)**.** *Suppose $\{x_n\}$ is a sequence of real numbers which is monotone, then $\{x_n\}$ is convergent if and only if $\{x_n\}$ is bounded.*

Consider the sequence $x_n = (-1)^n$. Although it is divergent, it is not far from being convergent. Namely, $x_{2n} = 1$ and $x_{2n+1} = -1$ for all n which are both convergent.

We need the concept of sub-sequence.

Definition 2.2. *Given a sequence of integer $n_1 < n_2 < ... < n_k < ...$, the sequence $\{x_{n_k}\}_{k=1}^{\infty}$ is said to be a sub-sequence of the original sequence $\{x_n\}$.*

Theorem 2.2. *Suppose $\{x_n\}$ is a convergent sequence, then any sub-sequence $\{x_{n_k}\}_{k=1}^{\infty}$ is convergent with the same limit.*

Using the terminology, we can state the definition of divergence by the following equivalent form.

Theorem 2.3. *Given a sequence $\{x_n\}$, then the following is equivalent:*

(1) *x_n is NOT convergent to x;*
(2) *$\exists \varepsilon_0 > 0$, and a subsequence $\{x_{n_k}\}$ such that for all k,*

$$|x_{n_k} - x| \geq \varepsilon_0$$

Moreover, the boundedness is almost equivalent to convergence in the following sense.

Theorem 2.4 (Bolzano-Weierstrass Theorem)**.** *Suppose $\{x_n\}$ is a bounded sequence, then there is a convergent subsequence.*

We will give an alternative proof which is different from that in textbook.

Proof. By boundedness, there is a, b such that for all n,

$$a \leq x_n \leq b.$$

For $k = 0$, we denote $I_0 = [a, b]$, $a_0 = a$ and $b_0 = b$. Suppose $[a, \dfrac{a_0 + b_0}{2}]$ contains infinity many x_k, then we choose $a_1 = a_0$, $b_1 = \frac{a_0+b_0}{2}$ otherwise we choose $a_1 = \frac{a_0+b_0}{2}$ and $b_1 = b_0$. Then we define $I_1 = [a_1, b_1]$ and pick $x_{n_1} \in I_1$. This is possible since I_1 contains infinity many elements.

We repeat the same step to obtain a sequence of I_k so that I_k is a sequence of closed, bounded and nested sequence. Moreover, there is $x_{n_k} \in I_k$ and

$$|I_k| = \frac{b-a}{2^k}.$$

By nested interval theorem, we have $\eta \in \cap_{k=1}^{\infty} I_k$. Therefore,

$$|\eta - x_{n_k}| \leq |I_k| = \frac{b-a}{2^k}$$

which implies $x_{n_k} \to \eta$ as $k \to +\infty$. $\square$

Math 2050, quick note of Week 5

1. Bolzano-Weiestrass Theorem

By boundedness Theorem, a convergent sequence must be bounded. It turns out to be almost equivalent statement!

Theorem 1.1 (Bolzano-Weiestrass Theorem)**.** *Suppose $\{x_n\}_{n=1}^\infty$ is a bounded sequence, then it admits a convergent sub-sequence.*

As a application,

Corollary 1.1. *If $\{x_n\}_{n=1}^\infty$ is bounded such that all convergent subsequence has the same limit, then $\{x_n\}_{n=1}^\infty$ is convergent with the same limt.*

2. Limit Superior and Limit Inferior

remark: I am not following the approach in textbook.

Recall that we only concern the behaviour when $n \to +\infty$. The convergence is equivalent to say that x_n is stabilized somewhere. To capture the "stability", it is often useful to consider the Oscillation of the tails.

Definition 2.1. *Given a bounded sequence $\{x_n\}_{n=1}^\infty$. Define*

(1)
$$\limsup_{n\to+\infty} x_n = \inf_{k\in\mathbb{N}} \sup_{n\geq k} x_n = \lim_{k\to+\infty} \sup_{n\geq k} x_n;$$
(2)
$$\liminf_{n\to+\infty} x_n = \sup_{k\in\mathbb{N}} \inf_{n\geq k} x_n = \lim_{k\to+\infty} \inf_{n\geq k} x_n.$$

Here the limits **Always** *exist by monotone convergence theorem. (1) capture the "max" of tail while (2) capture the "min".*

We have the equivalent form of definition (also equivalent to the one from the textbook).

Theorem 2.1. *Given a bounded sequence $\{x_n\}_{n=1}^\infty$, the followings are equivalent.*

1. $x = \limsup_{n\to+\infty} x_n$;
2. *For $\varepsilon > 0$, there are at most finitely many n such that $x+\varepsilon < x_n$ but infinity many n so that $x - \varepsilon < x_n$;*
3. $x = \inf V$ *where* $V = \{v \in \mathbb{R} : v < x_n \text{ for at most finitely many} n\}$;
4. $x = \sup S$ *where* $S = \{s \in \mathbb{R} : s = \lim_{k\to+\infty} x_{n_k} \text{ for some } \{n_k\}_{k=1}^\infty\}$.

Proof. (1) $\Rightarrow$ (2):

For all $\varepsilon > 0$, there is $k_0 \in \mathbb{N}$ such that for all $m \geq k > k_0$,

$$x + \varepsilon > \sup_{n \geq k} x_n \geq x_m.$$

Hence,

$$|\{i : x_i \leq x + \varepsilon\}| < +\infty$$

Moreover, $x - \varepsilon < \sup_{n \geq k} x_n$ for all $k \in \mathbb{N}$. Therefore, for each $k \in \mathbb{N}$, there is $n_k \geq k$ such that $x - \varepsilon < x_{n_k}$. Since $k \to +\infty$,

$$|\{i : x_i > x - \varepsilon\}| = +\infty.$$

(2) $\Rightarrow$ (3):

By (2), $x + \varepsilon \in V$ and hence $x + \varepsilon \geq \inf V$ for all $\varepsilon > 0$. By letting $\varepsilon \to 0$, we have

$$x \geq \inf V.$$

Suppose $x > \inf V$, there is $\varepsilon_0 > 0$ and $v \in V$ such that

$$x - \varepsilon_0 > v.$$

By (2) again, there are infinitely many x_n so that

$$x_n > x - \varepsilon > u$$

which contradicts with $u \in V$. Hence $x = \inf V$.

(3) $\Rightarrow$ (4): We claim something slightly stronger: $\inf V = \sup S$.

Let $v \in V$, since there are at most finitely many x_n such that $v < x_n$. There is $N \in \mathbb{N}$ such that for all $n > N$, $v \geq x_n$. Let $s \in S$, there is n_k such that $x_{n_k} \to s$. Applying the properties of v on x_{n_k}, we have for all $k > N$,$v \geq x_{n_k}$. Hence,

$$v \geq s.$$

The inequality is true for all $s \in S, v \in V$. Hence, $\inf V \geq \sup S$.

We now claim that $\inf V = \sup S$. If not, there is $\varepsilon_0 > 0$ such that

$$a = \inf V - \varepsilon_0 > \sup S.$$

There is $N \in \mathbb{N}$ such that for all $n > N$, $a > x_n$. Since otherwise, we can find a subsequence x_{n_k} such that $a \leq x_{n_k}$ for all k. By Bolzano-Weiestrass Theorem, there is $x_{n_{k_j}}$ which converges to some $s \in S$ as $j \to +\infty$ so that $a \leq s \leq \sup S$ which is impossible. Therefore,

$$|\{n : a < x_n\}| < +\infty$$

which implies $a \in V$ which is impossible.

(4) $\Rightarrow$ (1):

Let $s \in S$, there is $x_{n_k} \to s$. On the other hand, for all $k \in \mathbb{N}$,

$$\sup_{n \geq k} x_n \geq x_{n_k}.$$

By passing $k \to +\infty$, we have $\limsup_{n\to+\infty} x_n \geq s$ and hence

$$\limsup_{n\to+\infty} x_n \geq \sup S.$$

Denote $\bar{x} = \limsup_{n\to+\infty} x_n$. To show the opposite inequality, let $\varepsilon > 0$, we have for all $k \in \mathbb{N}$,

$$\bar{x} - \varepsilon < \sup_{n \geq k} x_n.$$

Therefore, for all $k \in \mathbb{N}$, there is x_{n_k} such that $\bar{x} - \varepsilon < x_{n_k}$. By Bolzano-Weiestrass Theorem, there is $x_{n_{k_j}} \to s$ for some $s \in S$ as $j \to +\infty$. This shows

$$\bar{x} - \varepsilon \leq s \leq \sup S, \quad \forall \varepsilon > 0.$$

By letting $\varepsilon \to 0$, we have

$$\bar{x} \leq \sup S.$$

This completes the proof. □

The importance of lim sup and lim inf is that they always exist (without checking anything!!!!).

Theorem 2.2. *Given a bounded sequence $\{x_n\}$, it is convergent if and only if*

$$\limsup_{n\to+\infty} x_n = \liminf_{n\to+\infty} x_n.$$

Proof. Suppose the sequence is convergent: $x_n \to x$ for some $x \in \mathbb{R}$. For all $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that

$$|x_n - x| < e.$$

And hence,for all $k > N$,

$$x - \varepsilon \leq \inf_{n \geq k} x_n \leq \sup_{n \geq k} x_n \leq x + \varepsilon.$$

Let $k \to +\infty$ and followed by $\varepsilon \to 0$, we have

$$x \leq \liminf_{n\to+\infty} x_n \leq \limsup_{n\to+\infty} x_n \leq x.$$

To prove the opposite direction, let x be the common limit. Then for all $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that for all $k > N$,

$$\sup_{n \geq k} x_n < x + \varepsilon, \quad \inf_{n \geq k} x_n > x - \varepsilon,$$

which shows that for all $n > N$,

$$x - \varepsilon < x_n < x + \varepsilon.$$

This completes the proof.

$\square$

Math 2050, summary of Week 6

1. Cauchy Sequence

Motivation: How to determine the convergence without discussion on the precise value of limit?

Definition 1.1. *A sequence* $\{x_n\}_{n=1}^{\infty}$ *is said to be Cauchy if* $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ *such that for all* $m, n > N$,

$$|x_m - x_n| < \varepsilon.$$

In other word, instead of controlling the oscillation around the "limit", we control the oscillation between elements!

Expectation: Cauchy sequence is equivalent to convergent sequence!

Recall that a convergent sequence is Necessarily bounded (Bounded Theorem)! And bounded sequence are "almost" convergent by Bolzano-Weierstrass Theorem. We first have:

Lemma 1.1. *A Cauchy sequence is bounded.*

(The proof is essentially the same with Bounded Theorem).

Theorem 1.1 (Cauchy Criterion)**.** *A sequence* $\{x_n\}_{n=1}^{\infty}$ *in* $\mathbb{R}$ *is Cauchy sequence if and only if it is convergent.*

Proof. If $\{x_n\}$ is convergent, then there is $x \in \mathbb{R}$ such that for all $\varepsilon > 0$, there is N such that for all $n > N$,

$$|x_n - x| < \varepsilon/2.$$

Hence, for all $m, n > N$, we have $|x_m - x_n| \leq |x_n - x| + |x_m - x| < \varepsilon$. Therefore it is Cauchy. This proved a easier direction.

For the opposite direction, suppose the sequence is Cauchy. Then it is bounded, hence there is $\{x_{n_k}\}_{k=1}^{\infty}$ such that $x_{n_k} \to x$ for some $x \in \mathbb{R}$ as $k \to +\infty$.Using Cauchy assumption, for all $\varepsilon > 0$, there is N such that for all $m, n > N$,

$$|x_m - x_n| < \varepsilon/2.$$

By replacing m by m_k with $k > N$, we have for all $k, n > N$,

$$|x_n - x_{m_k}| < \varepsilon/2.$$

Since this is true for all $k > N$, we may let $k \to +\infty$ to show

$$|x_n - x| \leq \varepsilon/2$$

for all $n > N$. This completes the proof.

$\square$

Math 2050, summary of Week 7

1. Series

Definition 1.1. *Given a sequence* $\{x_n\}_{n=1}^{\infty}$*, the series generated by* $\{x_n\}_{n=1}^{\infty}$ *is given by* $s_i = \sum_{k=1}^{i} x_k$*.*

Examples: The followings are the most important (and fundamental) examples of series.

(1) Geometric series: $\sum_{i=1}^{k} r^i$;
(2) Harmonic series $\sum_{n=1}^{k} n^{-1}$;
(3) p-series $\sum_{n=1}^{k} n^{-p}$;

Clearly, series is a special case of sequence. We are interested in their convergence since they are special and appear quite often.

We start with the elementary nature of series.

Theorem 1.1 (The n-the term test)**.** *Suppose* $\sum x_n$ *converges, then* $x_n \to 0$*.*

E.g. $\sum(-1)^n$ is divergent since $(-1)^n$ does not converge to 0. But it is far from equivalent. For example, $\sum k^{-1}$ is unbounded and divergent (as shown in previous lecture), but $k^{-1} \to 0$ as $k \to +\infty$.

Since the theory of sequence is better developed (at this stage), we now translate the corresponding Theorem in the setting of series.

Theorem 1.2 (Cauchy criterion)**.** *The series* $\sum x_n$ *is convergent if and only if* $\forall \varepsilon > 0, \exists N$ *such that for all* $m > n > N$*, we have*

$$|\sum_{k=n+1}^{m} x_k| < \varepsilon.$$

Theorem 1.3 (monotone convergence theorem)**.** *Suppose* $x_n \geq 0$ *for all* $n \in \mathbb{N}$*, then the series* $\sum x_n$ *is convergent if and only if there is* $M > 0$ *such that for all* $m \in \mathbb{N}$*,*

$$\sum_{k=1}^{m} x_k \leq M.$$

Example (Useful trick): $\sum k^{-2}$ is convergent. By MCT, it suffices to show the boundedness.

$$\begin{aligned}\sum_{k=1}^{m}\frac{1}{k^2} &\le 1+\sum_{k=2}^{m}\frac{1}{k^2}\\ &\le 1+\sum_{k=2}^{m}\frac{1}{k(k-1)}\\ &\le 1+\sum_{k=2}^{m}\left(\frac{1}{k-1}-\frac{1}{k}\right)\\ &\le 2-\frac{1}{m}\\ &< 2.\end{aligned} \tag{1.1}$$

And hence it is convergent.

This can be generalized further as one can see the argument only relies on some comparison after some large index.

Theorem 1.4 (Comparison Test)**.** *Suppose $\{x_n\}$ and $\{y_n\}$ are sequence such $0 \le x_n \le y_n$ for all $n > k_0$. Then we have*

(1) *$\sum x_n$ is convergent if $\sum y_n$ is convergent.*
(2) *$\sum y_n$ is divergent if $\sum x_n$ is divergent.*

Therefore, one only need to find some "reference" series to determine the convergence.

The convergence of the fundamental Examples:

(a) $\sum r^k$ is convergent if $r < 1$ and is divergent if $r \ge 1$;
(b) $\sum k^{-p}$ is convergent if $p > 1$ and is divergent if $p \le 1$.

2. Function

Let $A \subset \mathbb{R}$ and $f : A \to \mathbb{R}$ be a function on A.

Ultimate Objective: Study the continuity of f.

We are only interested in some "meaningful" point.

Definition 2.1. *Let $A \subset \mathbb{R}$. A real number c is said to be a cluster point of A if for all $\delta > 0$, there is $x \in A$ such that $0 < |x - c| < \delta$.*

It is easy to see that equivalently we can approximate c by sequence in $A \setminus \{c\}$.

Theorem 2.1. *Let $A \subset \mathbb{R}$. Then c is cluster point of A if and only if there is $\{a_n\} \subset A$ such that $a_n \neq c$ and $a_n \to c$ as $n \to +\infty$.*

Examples:

(1) $A = (0,1)$, then cluster points are $[0,1]$;
(2) $A = \{p_i\}_{i=1}^{k}$, then there are no cluster points;
(3) $A = \{k^{-1} : k \in \mathbb{N}\}$, then cluster point is $\{0\}$;
(4) $A = (0,1) \cap \mathbb{Q}$, then cluster points are $[0,1]$.

Cluster points are those points which is NOT isolated. Those are what we care!

Theorem 2.2. *Let $A \subset \mathbb{R}$ and c is a cluster point of A, $f : A \to \mathbb{R}$. Then $L \in \mathbb{R}$ is said to be the limit of f at c if for all $\varepsilon > 0$, there is $\delta > 0$, such that for all $x \in A$ with $0 < |x - c| < \delta$, we have $|f(x) - L| < \varepsilon$. In this case, we will denote $\lim_{x\to c} f = L$.*

The notion is reasonable since the limit is unique if exists.

Theorem 2.3. *Let $A \subset \mathbb{R}$ and c is a cluster point of A, $f : A \to \mathbb{R}$. Then f can at most have a single limit at c.*

The definition is usually not user friendly when we try to argue the opposite. We therefore need some alternative perspective of the definition.

Theorem 2.4 (Sequence criterion)**.** *Let $A \subset \mathbb{R}$ and c is a cluster point of A, $f : A \to \mathbb{R}$. Then we have $\lim_{x\to c} f = L$ if and only if for any sequence $\{a_n\} \subset A \setminus \{c\}$ so that $a_n \to c$, we have $f(a_n) \to L$.*

The contra-positive statement is given as follows.

Theorem 2.5 (Divergent criterion)**.** *Let $A \subset \mathbb{R}$ and c is a cluster point of A, $f : A \to \mathbb{R}$. Then*

(1) *f does not have a limit L at c if and only if $\exists \varepsilon_0 > 0$, $\{x_n\} \subset A \setminus \{c\}$ such that $x_n \to c$ but $|f(x_n) - L| \geq \varepsilon_0$ for all n.*
(2) *f does not have a limit at c if and only if $\exists \{x_n\} \subset A \setminus \{c\}$ such that $x_n \to c$ but $\{f(x_n)\}_{n=1}^{\infty}$ is divergent.*

Examples: *Direct application of the divergent criterion is to show that $\lim_{x\to 0} x^{-1}$ and $\lim_{x\to 0} \sin(x^{-1})$ does not exist.*

Math 2050, summary of Week 8

1. Ordering and convergence

We are always working on the following situation: $A \subset \mathbb{R}$ and c is a cluster point of A. We have known that the limit of function at cluster points have similar properties as the limit of sequence. We also have the following related to the ordering.

Theorem 1.1. *Let $f, g, h : A \to \mathbb{R}$, if*

$$f(x) \le g(x) \le h(x)$$

for all $x \in A$, then

(1) *if $\lim_{x\to c} f = F$, $\lim_{x\to c} g = G$ and $\lim_{x\to c} h = H$, we have*

$$F \le G \le H.$$

(2) *if $\lim_{x\to c} f = \lim_{x\to c} h = L$, then g has a limit as $x \to c$. In particular the limit is L.*

The importance of this result to that whenever we have well-behaved competitor, we can determine the convergence at some particular point.

Examples:

(1) $\lim_{x\to 0} \frac{\sin x}{x} = 1$ using $x - \frac{1}{6}x^3 \le \sin x \le x$ for all $x \ge 0$.
(2) $\lim_{x\to 0} \frac{\cos x - 1}{x} = 0$ using $-\frac{1}{2}x^2 \le \cos x - 1 \le 0$ for $x > 0$.
(3) $\lim_{x\to 0} x\sin(1/x) = 0$ using $|x \sin x| \le |x|$.

2. Some variation of limits

2.1. **One sided limits.** Consider the example

$$f(x) = \begin{cases} e^{1/x}, & \text{if } x \ne 0; \\ 0, & \text{if } x = 0; \end{cases} \tag{2.1}$$

Then the function has different behaviour when x tends to 0 from different directions. It is sometimes more important to consider one particular direction rather than all directions.

Definition 2.1. *Let $A \subset \mathbb{R}$ and c is a cluster point of $A \cap (c, +\infty)$, $f : A \to \mathbb{R}$. We say that $\lim_{x\to c^+} f = L$ if $\forall \varepsilon > 0, \exists \delta > 0$ such that if $x \in A$ where $0 < x - c < \delta$, then*

$$|f(x) - L| < \varepsilon.$$

(similar for the left hand limit)

As expected from the theory for function, we have the following characterization using sequence which will be more user friendly when we discuss the divergence.

Theorem 2.1. *Let $A \subset \mathbb{R}$ and c is a cluster point of $A \cap (c, +\infty)$, $f : A \to \mathbb{R}$. Then $\lim_{x \to c^+} f = L$ if and only if for any $x_n \in A \cap (c, +\infty)$ where $x_n \to c$, we have $f(x_n) \to L$.*

Example: The function f defined at the beginning have $\lim_{x \to 0^-} f = 0$ but has no limit as $x \to 0^+$.

2.2. **Infinite limit.** The behaviour of f defined above as $x \to 0^+$ is divergent, but its divergence is relatively well-behaved as $f(x) \to +\infty$.

Definition 2.2. *Let $A \subset \mathbb{R}$ and c is a cluster point of A, $f : A \to \mathbb{R}$. We say that $\lim_{x \to c} f = +\infty$ if $\forall \alpha > 0$, there is $\delta > 0$ such that if $x \in A$ such that $0 < |x - c| < \delta$, then $f(x) > \alpha$.*

One might compare this with the sequence.

Definition 2.3. *A sequence $\{a_n\}$ is said to be divergent to $+\infty$ if $\forall \alpha > 0, \exists N \in \mathbb{N}$ such that for all $n > N$, $a_n > \alpha$.*

The sequence criterion can be formulated similarly.

Theorem 2.2. *Let $A \subset \mathbb{R}$ and c is a cluster point of A, $f : A \to \mathbb{R}$. Then $\lim_{x \to c} f = +\infty$ if and only if for any $x_n \in A \setminus \{c\}$ where $x_n \to c$, we have $f(x_n) \to +\infty$.*

2.3. **limit at infinity.** The example mentioned above has certain decay properties as $x \to \infty$. To make it rigorous, we have

Definition 2.4. *Let $A \subset \mathbb{R}$ and suppose $(a, +\infty) \subset A$ for some $a \in \mathbb{R}$, and $f : A \to \mathbb{R}$. We say that $\lim_{x \to +\infty} f = L$ if $\forall \varepsilon > 0$, there $\alpha \in \mathbb{R}$ such that for all $x > \alpha$,*

$$|f(x) - L| < \varepsilon.$$

Similarly, we have

Definition 2.5. *Let $A \subset \mathbb{R}$ and suppose $(a, +\infty) \subset A$ for some $a \in \mathbb{R}$, and $f : A \to \mathbb{R}$. We say that $\lim_{x \to +\infty} f = +\infty$ if $\forall \beta > 0$, there $\alpha \in \mathbb{R}$ such that for all $x > \alpha$,*

$$f(x) > \beta.$$

Example:

(1) $\lim_{x \to +\infty} x^m = +\infty$ for all $m \in \mathbb{N}$;
(2) $\lim_{x \to +\infty} p(x) = +\infty$ if $p(x) = \sum_{i=0}^n a_i x^i$ where $a_n > 0$;
(3) $\lim_{x \to +\infty} e^{-x} = 0$.

3. Continuous function

Recall that to consider the limit $\lim_{x\to c} f = L$, we allow the situation:

$$f(x) = \begin{cases} x, & \text{if } x \neq 0; \\ 1, & \text{if } x = 0; \end{cases} \tag{3.1}$$

It is clear that $f(0) \neq \lim_{x\to 0} f$. But the limit is still well-behaved. (That is why we only consider $x \in A, 0 < |x - c| < \delta$ in the definition without considering $x = c$.)

We now pay more attention to the case when f is continuous. Want to rule out the above situation!

Definition 3.1. *Let $A \subset \mathbb{R}, c \in A$ and $f : A \to \mathbb{R}$. We say that f is continuous at c if $\forall \varepsilon > 0, \exists \delta > 0$ such that for all $x \in A$, $|x - c| < \delta$, we have*

$$|f(x) - f(c)| < \varepsilon.$$

Remark:

(1) If c is a cluster point, then the continuity implies i) $c \in A$; ii) f has limit at c; iii)$\lim_{x\to c} f = f(c)$.
(2) if c is not a cluster point, then there is $\delta > 0$ such that $A \cap \{y : 0 < |y - c| < \delta\} = \emptyset$. Hence, we always have the continuity f at c.

In term of sequence criterion:

Theorem 3.1. *Let $A \subset \mathbb{R}$ and $c \in A$, $f : A \to \mathbb{R}$. Then $\lim_{x\to c} f = f(c)$ if and only if for any $x_n \in A$ where $x_n \to c$, we have $f(x_n) \to f(c)$.*

Example:

(a)

$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q}; \\ 0, & \text{otherwise.} \end{cases} \tag{3.2}$$

Then f is discontinuous at any point $c \in \mathbb{R}$. If $c \in \mathbb{Q}$, then $f(c) = 1$. But we can take $x_n \notin \mathbb{Q}$ by density of $\mathbb{Q}'$ such that $x_n \to c$ so that $f(x_n) \equiv 0$. Similarly, if $c \notin \mathbb{Q}$, we have similar contradicting sequence.

(b)

$$f(x) = \begin{cases} \frac{1}{n}, & \text{if } x = \frac{m}{n} \text{ where } gcd(m, n) = 1; \\ 0, & \text{otherwise.} \end{cases} \tag{3.3}$$

Then f is discontinuous at $c \in \mathbb{Q}$ and is continuous at $c \notin \mathbb{Q}$. Discontinuous is similar to the first example. We may assume $c > 0$.

To show the continuity, for $\varepsilon > 0$, fix N such that $N^{-1} < \varepsilon$. Consider the set $\{(m, n) : n \leq N\}$, if $|mn^{-1} - c| < c$, we have

$$m \leq 2cn \leq 2cN.$$

Therefore, there is only finitely many element in form of $x = mn^{-1}$ so that $|x - c| < 1$ and $n \leq N$. Since $c \notin \mathbb{Q}$, c is not one of the element. Hence, c is isolated from the set $\{mn^{-1} : n \leq N\} \cap \{x : |x - c| < 1\}$. Therefore, we can find $\delta > 0$ such that

$$B = \{x : |x - c| < \delta\} \cap \{mn^{-1} : n \leq N\} = \emptyset.$$

Hence, for $x \in B$, we have $f(x) = \frac{1}{n} < \frac{1}{N} < \varepsilon$.

Math 2050, summary of Week 9-10

Recall the definition of continuity of a function.

Definition 0.1. *A function $f : A \to \mathbb{R}$ is said to be continuous at $c \in A$ if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that if $|x-c| < \delta, x \in A$, then $|f(x) - f(c)| < \varepsilon$. The function f is said to be continuous on A, if f is continuous at $c \in A$ for all $c \in A$.*

Important remark: the choice of δ is a-priori depending on the point $c \in A$.

Example: $\lim_{x \to c} x^n = c^n$ for any given $n \in \mathbb{N}$.

Proof. We first consider the error:

$$
\begin{aligned}
|f(x) - f(c)| &= |x^n - c^n| \\
&= |x-c| \left| \sum_{k=0}^{n-1} x^{n-1-k} c^k \right| \\
&\le |x-c| \left(\sum_{k=0}^{n-1} |x|^{n-1-k} |c|^k \right).
\end{aligned} \tag{0.1}
$$

Like before, it suffices to control the "coefficient". We fix $\delta = \min\{\Lambda\varepsilon, 1\}$ where we will specify Λ later. Then if $|x-c| < \delta \le 1$, we have

$$
|x|^{n-1-k} \le (|x-c| + |c|)^{n-1-k} \le (1 + |c|)^{n-1-k}. \tag{0.2}
$$

Hence,

$$
\begin{aligned}
|f(x) - f(c)| &\le |x-c| \left(\sum_{k=0}^{n-1} (1+|c|)^{n-1-k} |c|^k \right) \\
&= M_c |x-c|
\end{aligned} \tag{0.3}
$$

where M_c is the number depending on the value of c. Then by choosing $\Lambda = M^{-1}$ which also depends on c, we have if $|x-c| < \delta$,

$$
|f(x) - f(c)| < \varepsilon.
$$

In this way, it is clear that the choice of δ is possibly depending also on the given point! This (in)dependence will be important later! □

Some algebra of continuity (using sequence criterion):

Theorem 0.1. *Let $A \subset \mathbb{R}$ and $f, g : A \to \mathbb{R}$ be functions continuous at $c \in A$ and $\lambda \in \mathbb{R}$. Then $f+g, f-g, \lambda f, fg$ are continuous at $c \in A$. If $g(x) \neq 0$ on A, then fg^{-1} is continuous at $c \in A$.*

As a immediate applications: polynomials are continuous on $\mathbb{R}$.

More properties of continuous functions (also using sequence criterion):

Theorem 0.2. *Let $A, B \subset \mathbb{R}$, $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ such that $f(A) \subset B$. If f is continuous at $c \in A$ and g is continuous at $f(c)$, then $g \circ f$ is continuous at $c \in A$.*

Proof. Let $x_n \in A$ such that $x_n \to c$. Since f is continuous at c, we have $f(x_n) \to f(c)$. Using sequence criterion again, since g is continuous at $f(c)$ and $f(x_n) \to f(c)$, we have $g(f(x_n)) \to g(f(c))$. Since x_n is arbitrary, we have the continuity of $g \circ f$ at $c \in A$. $\square$

Example: $f(x) = \sqrt{x + \sqrt{x}}, \sin|x|, etc$ are continuous on $\mathbb{R}^+$.

1. Continuous functions on closed and bounded intervals

Examples:

(1) $f(x) = x^{-1}$ on $(0, 1]$;
(2) $f(x) = (x + 1)^{-1}$ on $[0, 1]$.

If we allow the interval to be open, the first example states that we allow the function badly behaved nearby boundary even if we impose continuity (since this is local information). But if the function is continuous on a closed and bounded interval, the structure of domain limits the possibility of bad behavior. The second function is bounded. And this is true in general.

Theorem 1.1. *Suppose $f : [a, b] \to \mathbb{R}$ is continuous, then there is $M > 0$ such that $|f(x)| \le M$ for all $x \in [a, b]$.*

Remark 1.1. One might compare the local boundedness theorem in previous lecture: If f is continuous at $c \in A$, then there is $\delta_c, M_c > 0$ such that $|f(x)| \le M_c$ for all $x \in A, |x-c| < \delta_c$. This local boundedness theorem doesn't imply the global boundedness as can be seen from the example $f(x) = x^{-1}$. This is because the δ_c found using continuity depends on the center c. As $c \to \partial A$, δ_c might degenerate, and M_c might blow up to $+\infty$ which gives us no information. (Think about the explicit value of δ_c in the example $f(x) = x^{-1}$ on $(0, 1]$).

Remark 1.2. As mentioned above, the assumption of closeness is necessary, $f(x) = x^{-1}$ on $(0, 1]$ is unbounded but is continuous. The boundedness is also necessary, as can be seen from $f(x) = x$ on $[0, +\infty)$.

Thanks to the boundedness Theorem, it is clear from completeness axiom that both

$$(1.1) \qquad M = \sup\{f(x) : x \in [a,b]\}, \quad m = \inf\{f(x) : x \in [a,b]\}$$

exists as a real number. The next Theorem shows that m, M can in fact be achieved.

Theorem 1.2 (Max-Min Theorem)**.** *Suppose $f : [a,b] \to \mathbb{R}$ is a continuous function, then there is $x_1, x_2 \in [a,b]$ such that $f(x_1) = M$ and $f(x_2) = m$ so that for all $x \in [a,b]$,*

$$f(x_1) \le f(x) \le f(x_2).$$

Sketch of Proof. (Refer to Textbook if you wish more detail) By definition of sup, take x_i such that $f(x_i) \to M$. Since $x_i \in [a,b]$, $x_{i_k} \to \bar{x}$ for some $\bar{x} \in [a,b]$. By Sequence criterion, $f(x_{i_k}) \to f(\bar{x}) = M$. The lower bound is similar. □

Some variation of Max-Min Theorem: Given a continuous function $f : [a,b] \to \mathbb{R}$. How can we find $\bar{x}$ such that $f(\bar{x}) = 0$? First of all, if $f(x) > 0$ or < 0 for all x, this is clearly impossible. If $f \equiv 0$, then the assertion is trivial. What if f is positive and negative somewhere?

Theorem 1.3. *Suppose $f : [a,b] \to \mathbb{R}$ is a continuous such that $f(a) > k > f(b)$ for some $k \in \mathbb{R}$, then there is $\bar{x} \in [a,b]$ such that $f(\bar{x}) = k$.*

Proof. By translation, we may assume $k = 0$. In the textbook (or in class), we use the bisection method which is a algorithm to locate the root. Here I am going to give an alternative proof (also discussed in class).

Let $p = \sup S = \sup\{s \in [a,b] : f(x) > 0 \ \ \forall x \in [a,s]\}$. Since $a \in S$, $p \in \mathbb{R}$ exists by completeness. It suffices to show that $f(p) = 0$, namely is the first root. Clearly, $p > a$ by using the continuity at a.

Assume $f(p) > 0$, then there is $\delta > 0$ such that for all $x \in (p-\delta, p+\delta) \subset [a,b]$, we have $f(x) > 0$.

Since $p - \delta < p = \sup S$, there is $s_0 \in S$ such that $p - \delta < s_0$ and hence we $f(x) > 0$ on $[a, p+\delta)$. This implies $p + \delta/2 \in S$ which is impossible.

Assume $f(p) < 0$, then similarly there is $\delta > 0$ such that for all $x \in (p - 2\delta, p + 2\delta) \subset [a,b]$, we have $f(x) < 0$. By the same argument, we have $f(x) > 0$ on $[a, p-\delta]$ which is impossible.

Therefore, we must have $f(p) = 0$! □

As an immediate application, we have

Corollary 1.1. *If* $f : [a, b] \to \mathbb{R}$ *is a continuous function, then* $f([a, b])$ *is a closed and bounded interval.*

Math 2050, summary of Week 11

1. Continuous on closed and bounded interval, cont.

Recall that if the domain is closed and bounded, the behavior of the functions can't be too bad due to the constraints on the boundary (roughly speaking). In case $f : \mathbb{R} \to \mathbb{R}$ is continuous, then the "boundary" is referring to the infinity ∞. Oppositely, if the behavior at ∞ is well-controlled, then the function is also well-controlled globally to some extent.

Example. If $p(x)$ is a polynomial of odd degree, then $p(x)$ has at least one real root.

Sketch of Proof in the lecture. Denote $p(x) = \sum_{i=0}^{2n+1} a_i x^i$. We may assume $a_{2n+1} > 0$, otherwise consider $-p(x)$. By the definition of limit, there is M such that for $|x| \geq M$,

$$\frac{p(x)}{a_{2n+1}x^{2n+1}} \geq \frac{1}{2}.$$

In particular, if $x \geq M$, then $p(x) > 0$ while if $x \leq -M$, $p(x) < 0$. The result follows from applying intermediate value theorem on $[-M, M]$. □

Example. If $f : \mathbb{R} \to \mathbb{R}$ is continuous such that $\lim_{x\to+\infty} f(x) = \alpha$ and $\lim_{x\to-\infty} f(x) = \beta$ for some $\alpha, \beta \in \mathbb{R}$, then f is bounded on $\mathbb{R}$.

Sketch of Proof. By using definition of limit, there is M such that if $|x| \geq M$, we have

$$|f(x)| \leq |\alpha| + |\beta| + 1.$$

Then applying boundedness Theorem on $[-M, M]$, there is Λ such that for all $|x| \leq M$, $|f(x)| \leq \Lambda$. Combines two inequalities, we obtain an upper bound on $\mathbb{R}$.

□

2. Uniform continuity

Example. $f(x) = x^{-1}$ on $(0, 1)$. Recall that: To examine the continuity, we choose δ in the following way. For $c \in (0, 1)$,

$$|f(x) - f(c)| = \frac{1}{xc}|x - c|. \tag{2.1}$$

Hence, for $\varepsilon > 0$, we choose $\delta_c = \min\{\frac{c}{2}, \frac{1}{2}c^2\varepsilon\}$ so that for $|x - c| < \delta$, we have $|f(x) - f(c)| < \varepsilon$. In this way, δ_c depends on the choice of c

and more importantly, $\delta_c \to 0$ as $c \to 0$. This is the reason why we can't obtain uniform boundedness for f in this example!

To overcome this, we introduce a new concept.

Definition 2.1. *Let $A \subset \mathbb{R}$ and $f : A \to \mathbb{R}$. We say that f is uniformly continuous on A if $\forall \varepsilon > 0$, there is $\delta > 0$ such that if $x, y \in A$ and $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon$.*

Clearly, an uniform continuous function is continuous. More importantly, the uniform continuity depends on the domain!

Theorem 2.1. *Let $A \subset \mathbb{R}$ and $f : A \to \mathbb{R}$. The function f is NOT uniformly continuous if and only if $\exists \varepsilon_0 > 0$ and two sequence $\{x_n\}_{n=1}^\infty, \{y_n\}_{n=1}^\infty \subset A$ such that $|x_n - y_n| \to 0$ but $|f(x_n) - f(y_n)| \geq \varepsilon_0$ for all n.*

Example. $f(x) = x^{-1}$ is uniformly continuous on $(a, +\infty)$ if $a > 0$. And f is NOT uniformly continuous if $a = 0$.

Proof. We compute

$$|f(x) - f(c)| = \frac{1}{xc}|x - c| \leq a^{-2}|x - c|. \tag{2.2}$$

Hence, $\forall \varepsilon > 0$, $\exists \delta = a^2 \varepsilon$ so that if $|x - c| < \delta$ and $x, c > a$, we have $|f(x) - f(c)| < \varepsilon$.

If $a = 0$, the proof fails (but this is not a proof!). To show the non-uniform continuity, we choose $x_n = n^{-1}$, $y_n = 2n^{-1}$ so that $|x_n - y_n| \leq n^{-1} \to 0$ but

$$|f(x_n) - f(y_n)| = n - \frac{n}{2} = \frac{n}{2} \geq \frac{1}{2}. \tag{2.3}$$

□

Theorem 2.2. *Suppose $f : (a, b) \to \mathbb{R}$ is uniformly continuous, then f is bounded.*

Proof. Let $\varepsilon = 1$, there is $\delta > 0$ such that if $|x - y| < \delta$ and $x, y \in (a, b)$, then

$$|f(x) - f(y)| < 1.$$

Let N be large enough so that $(b - a)/N < \delta$ and define $x_n = a + nN^{-1}(b - a)$ so that if $x \in (a, b)$, then $|x - x_n| < \delta$ for some $n = 1, 2, ..., N$. Hence,

$$\begin{aligned} |f(x)| &\leq |f(x) - f(x_n)| + |f(x_n)| \\ &\leq 1 + \max\{|f(x_i)| : i = 1, ..., N\} \\ &= M. \end{aligned} \tag{2.4}$$

3

□

Math 2050, summary of Week 11

1. Uniform continuity

Recall that $f : A \to \mathbb{R}$ is said to be uniform continuous if $\forall \varepsilon > 0$, there is $\delta > 0$, such that for all $x, y \in A$ with $|x - y| < \delta$, we have $|f(x) - f(y)| < \varepsilon$.

In contrast with continuity of a function, *Uniform continuity is a global properties*!

Question: How to improve from continuity to uniform continuity?

Theorem 1.1. *If $f : [a, b] \to \mathbb{R}$ is continuous, then f is uniformly continuous.*

We here present an alternative proof different from that in the textbook. (more complicated but more intuitive)

Proof. Let $\varepsilon > 0$ be given. We consider the following subset S of $[a, b]$:

$$S = \{c \in [a, b] : \ \exists \delta > 0, \text{ such that } \forall x, y \in [a, c], |x-y| < \delta, \Longrightarrow |f(x)-f(y)| < \varepsilon\}.$$

Clearly, $a \in S$ and S is bounded from above. Hence, $s = \sup S \leq b$ exists. We will show that $s = b \in S$. Suppose not, $s < b$. By continuity of f, there is $\delta_s > 0$ such that for all $x \in [a, b]$ and $|x - c| < \delta_s$, we have

$$|f(x) - f(c)| < \frac{1}{2}\varepsilon.$$

On the other hand, since $s - \frac{1}{2}\delta_s < s$, there is $c \in S$ such that $s - \frac{1}{2}\delta_s \leq c$ and hence, we can find $\delta_c > 0$ so that for all $x, y \in [a, c]$ with $|x - y| < \delta_c$, we have

$$|f(x) - f(y)| < \varepsilon.$$

Now we choose $\delta = \min\{\delta_c, \frac{1}{4}\delta_s\} > 0$. For $x, y \in [a, s + \frac{1}{4}\delta_s] \cap [a, b]$ with $|x - y| < \delta$.

If $x, y \leq s - \frac{1}{2}\delta_s$, then we must have $|f(x) - f(y)| < \varepsilon$ since $|x - y| < \delta_c$. If $x \leq s - \frac{1}{2}\delta_s < y$, we have $|x - s|, |y - s| < \delta_s$ since $|x - y| < \frac{1}{4}\delta_s$, therefore

$$|f(x) - f(y)| \leq |f(x) - f(c)| + |f(y) - f(c)| < \varepsilon. \tag{1.1}$$

If $s - \frac{1}{2}\delta_s < x, y < s + \frac{1}{2}\delta_s \leq b$, then the same argument shows that $|f(x) - f(y)| < \varepsilon$. Hence, $s + \frac{1}{2}\delta_s \in S$ which is impossible. This shows that $b = s$. Moreover, the same argument shows that $s \in S$. This completes the proof. ∎

From the Theorem, the continuity on closed and bounded interval is automatically uniform continuous. The inverse is also true in a suitable sense.

Theorem 1.2. *Suppose $f:(a,b)\to\mathbb{R}$ is uniformly continuous, then there is $\tilde{f}:[a,b]\to\mathbb{R}$ such that $\tilde{f}=f$ on (a,b) and is continuous on $[a,b]$.*

Proof. It suffices to show that $\lim_{x\to a^+} f(x)$ and $\lim_{x\to b^-} f(x)$ exist. Let $x_n\to a^+$. Since a uniformly continuous function on bounded interval must be bounded, $\{f(x_n)\}$ is a bounded sequence. Therefore, there is $x_{n_k}\to a$ such that $f(x_{n_k})\to L\in\mathbb{R}$ for some L.

Suppose $\lim_{x\to a^+} f(x)$ doesn't exists. Then for $L\in\mathbb{R}$, we can find $y_n\to a^+$ such that $f(y_n)\to l\in\mathbb{R}$ but $l\neq L$. But this contradicts with the uniform continuity as both x_n and y_n converges to a.

□