Assignment 4, Due 3/11/2022 on or before 11:59 pm

Please upload your assignment to the Blackboard of this course

(1) Let $M = \{(x, y, z)|\ z = x^2 + ky^2\}$, with $k > 0$. Show that $e_1 = (1, 0, 0)$ and $e_2 = (0, 1, 0)$ form a basis of $T_p(M)$ where $p = (0, 0, 0)$. Let $\mathbf{N}$ be the unit normal of M pointing upward, i.e. $\langle \mathbf{N}, e_3 \rangle > 0$ where $e_3 = (0, 0, 1)$. Find the matrix of $\mathcal{S}_p : T_p(M) \to T_p(M)$ with respect to the ordered basis $\{e_1,\ e_2\}$. Find the principal curvatures of M at p.

(2) Let $\lambda_1, \lambda_2, \cdots, \lambda_m$ be the normal curvatures at a point $p \in M$ along direction making angles $0, 2\pi/m, \cdots, (m-1)2\pi/m$ with a principal directions. Show that

$$\lambda_1 + \cdots + \lambda_m = mH$$

where H is the mean curvature at p.

(See problem 18, p.153 in Do Carmo's book).

(3) Suppose a regular surface M is tangent to a plane at a point. Moreover, M lies on one side of the plane. What can you say about the Gaussian curvature $K(p)$ of M at p? Is it possible that $K(p) < 0$?

(4) Show that the helicoid:

$$\mathbf{X}(u, v) = (a \sinh v \cos u, a \sinh v \sin u, au),$$

and the Enneper's surface

$$\mathbf{X}(u, v) = \left(u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + vu^2, u^2 - v^2 \right).$$

are minimal surfaces.

(5) A surface of revolution M parametrized by $\mathbf{X}(u, v) = (u, h(u) \cos v, h(u) \sin v)$. Find the mean curvature H of the surface. Suppose $H = \frac{c}{2} \neq 0$. Show that

$$h^2 + \frac{2ah}{\sqrt{1 + (h')^2}}$$

is a constant where $a = -\frac{1}{c}$.

(See p.134 in Oprea's book).