Normal curvatures

Let M be an orientable regular surface with an orientation $\mathbf{N}$. Let $\alpha(s)$ be a smooth curve on M parametrized by arc length.
Let $T = \alpha'$ and let $\mathbf{n}(s)$ be the unit vector at $\alpha(s)$ such that $\mathbf{n} \in T_{\alpha(s)}(M)$ and such that $\{T, \mathbf{n}, \mathbf{N}\}$ is positively oriented, i.e. $\mathbf{n} = \mathbf{N} \times T$.

Lemma

T' is a linear combination of $\mathbf{n}$ and $\mathbf{N}$: $T' = k_g \mathbf{n} + k_n \mathbf{N}$ for some smooth functions k_n and k_g on $\alpha(s)$.

Definition

As in the lemma, $k_n(s)$ is called the normal curvature of α at $\alpha(s)$ and $k_g(s)$ is called the geodesic curvature of α at $\alpha(s)$.

Facts:

- k_n and k_g depend on the choice of $\mathbf{N}$.
- We will see later that k_g is intrinsic: it depends only on the first fundamental form *and* the orientation of the surface.
- Let κ be the curvature of α'. Suppose κ is not zero. Let N_α be the normal of α (recalled $\alpha'' = \kappa N_\alpha$). Then $k_n = \kappa\langle N_\alpha, \mathbf{N}\rangle = k\cos\theta$ where θ is the angle between N and $\mathbf{N}$. If $k = 0$, then $T' = 0$ and $k_n = k_g = 0$.

Normal curvatures and second fundamental form

We first discuss normal curvature. Its relation with the the second fundamental form is the following:

Proposition

Let M be an orientable regular surface with an orientation $\mathbf{N}$. Let $\mathbb{II}$ be the second fundamental form of M (w.r.t. $\mathbf{N}$) and let $p \in M$. Suppose $\mathbf{v} \in T_p(M)$ with unit length and suppose $\alpha(s)$ is a smooth curve of M parametrized by arclength with $\alpha(0) = p$ and $\alpha'(0) = \mathbf{v}$. Then

$$k_n(0) = \mathbb{II}_p(\mathbf{v}, \mathbf{v})$$

where k_n is the normal curvature of α at $\alpha(0) = p$.

Proof.

$\mathcal{S}_p(\mathbf{v}) = -\frac{d}{ds}\mathbf{N}(\alpha(s))|_{s=0}$. Hence

$$\begin{aligned}\mathbb{II}_p(\mathbf{v},\mathbf{v}) &= \langle \mathcal{S}_p(\mathbf{v}), \mathbf{v}\rangle \\ &= \langle \mathbf{N}(\alpha(s)), \frac{d}{ds}\alpha' \rangle|_{s=0} \\ &= k_n(0).\end{aligned}$$

□

Corollary

With the same notation as in the proposition, we have the following: Let α and β be two regular curves parametrized by arc length passing through p. Suppose α and β are tangent at p. Then the normal curvatures of α and β at p are equal.

Basic facts on symmetric bilinear form

Let $(V, \langle \cdot\ ,\ \cdot \rangle)$ be a finite dimensional inner product space and let B be a *symmetric* bilinear form on V.

- Let Q be the corresponding quadratic form, $Q(\mathbf{v}) = B(\mathbf{v}, \mathbf{v})$
- A be the corresponding self-adjoint operator: $\langle A(\mathbf{v}), \mathbf{w} \rangle = B(\mathbf{v}, \mathbf{w})$.

Theorem

Let $(V, \langle \cdot\ ,\ \cdot \rangle)$ be a finite dimensional inner product space of dimension n and let B be a symmetric bilinear form. Then there is an orthonormal basis $\mathbf{v}_1, \ldots, \mathbf{v}_n$ such that B is diagonalized. Namely, $B(\mathbf{v}_i, \mathbf{v}_j) = \lambda_i \delta_{ij}$. $\mathbf{v}_i$ is an eigenvector of A with eigenvalue λ_i: $A(\mathbf{v}_i) = \lambda_i \mathbf{v}_i$. Moreover, if $\mathbf{v} = \sum_{i=1}^n x^i \mathbf{v}_i$, then $Q(\mathbf{v}) = \sum_{i=1}^n \lambda_i (x^i)^2$.

Proof

: We just prove the case that $n = 2$. Let S be the set in V with $||\mathbf{v}||^2 = \langle \mathbf{v}, \mathbf{v} \rangle = 1$. Then $B(\mathbf{v}, \mathbf{v})$ attains maximum on S at some $\mathbf{v}$. Let $\mathbf{v}_1 \in S$ be such that

$$B(\mathbf{v}_1, \mathbf{v}_1) = \max_{\mathbf{v} \in S} B(\mathbf{v}, \mathbf{v}).$$

Let $\mathbf{v}_2 \in S$ such that $\mathbf{v}_1 \perp \mathbf{v}_2$. It is sufficient to prove that $B(\mathbf{v}_1, \mathbf{v}_2) = 0$. Let $t \in \mathbb{R}$ and let

$$f(t) = \frac{B(\mathbf{v}_1 + t\mathbf{v}_2, \mathbf{v}_1 + t\mathbf{v}_2)}{||\mathbf{v}_1 + t\mathbf{v}_2||^2}.$$

Then $f'(0) = 0$. Hence

$$\begin{aligned} 0 =& 2B(\mathbf{v}_1, \mathbf{v}_2) - 2B(\mathbf{v}_1, \mathbf{v}_1)\langle \mathbf{v}_1, \mathbf{v}_2 \rangle \\ =& 2B(\mathbf{v}_1, \mathbf{v}_2). \end{aligned}$$

Let $\lambda_2 = B(\mathbf{v}_2, \mathbf{v}_2)$.
Now $\langle A(\mathbf{v}_1), \mathbf{v}_1\rangle = B(\mathbf{v}_1, \mathbf{v}_1) = \lambda_1 = \lambda_1\langle \mathbf{v}_1, \mathbf{v}_1\rangle$;
$\langle A(\mathbf{v}_1), \mathbf{v}_2\rangle = B(\mathbf{v}_1, \mathbf{v}_2) = 0 = \langle \mathbf{v}_1, \mathbf{v}_2\rangle$. Hence

$$\langle A(\mathbf{v}_1) - \lambda_1 \mathbf{v}_1, \mathbf{v}_i\rangle = 0$$

for $i = 1, 2$. Hence $A(\mathbf{v}_1) = \lambda_1 \mathbf{v}_1$.
Let $\mathbf{v} = \sum_{i=1}^n x^i \mathbf{v}_i$, then

$$\begin{aligned} Q(\mathbf{v}) =& B(\mathbf{v}, \mathbf{v}) \\ =& \sum_{i,j=1}^n x^i x^j B(\mathbf{v}_i, \mathbf{v}_j) \\ =& \sum_{i=1}^n \lambda_i (x^i)^2. \end{aligned}$$

Principal curvatures

Let M be an orientable regular surface with orientation $\mathbf{N}$.

Definition

Let $\mathbf{e}_1, \mathbf{e}_2$ be an orthonormal basis on $T_p(M)$ which diagonalizes $\mathbb{II}_p$ with eigenvalues k_1 and k_2. Then k_1, k_2 are called the principal curvatures of M at p and $\mathbf{e}_1, \mathbf{e}_2$ are called the principal directions. Suppose $k_1 \leq k_2$ then all normal curvature k must satisfies $k_1 \leq k \leq k_2$.

Principle curvatures and Gaussian curvature, mean curvature

Proposition

With the above notations, if $k_1 = k_2 = k$, then every direction is a principal direction and in this case, $\mathcal{S}_p = k\mathbf{id}$. (In this case, the point is said to be umbilical.) Moreover, the Gaussian curvature and the mean curvature are given by $K(p) = k_1 k_2$, and $H(p) = \frac{1}{2}(k_1 + k_2)$.

Regular surface where all points are umbilical

Proposition

Let $\mathbf{X} : U \to \mathbb{R}^3$ be an orientable regular surface, which is connected. Suppose every point in M is umbilical. Then M is contained in a plane or in a sphere.

Proof: Let us first consider a coordinate patch, $\mathbf{X}(u, v)$ with $(u, v) \in U$ which is connected. Let $\mathbf{N}$ be a unit normal vector field on M and let $\mathcal{S}$ be the shape operator. Then $\mathcal{S}_p(\mathbf{v}) = \lambda \mathbf{v}$ for any $\mathbf{v} \in T_p(M)$ for some function $\lambda(p)$. We write $\lambda = \lambda(u, v)$. This is smooth function. Now

$$-\mathbf{N}_u = \mathcal{S}_p(\mathbf{X}_u) = \lambda \mathbf{X}_u.$$

Hence $-\mathbf{N}_{uv} = \lambda_v \mathbf{X}_u + \lambda \mathbf{X}_{uv}$. Similarly, $-\mathbf{N}_{vu} = \lambda_u \mathbf{X}_v + \lambda \mathbf{X}_{vu}$. Hence $\lambda_u = \lambda_v = 0$ everywhere (Why?). So λ is constant in this coordinate chart. Hence λ is constant on M. (Why?).

Proof, cont.

Case 1: $\lambda \equiv 0$. Then $\mathbf{N}_u = \mathbf{N}_v = 0$. So $\mathbf{N} = \mathbf{a}$, which is a constant vector. Then

$$\langle \mathbf{X}(u,v) - \mathbf{X}(u_0,v_0), \mathbf{N}\rangle_u = \langle \mathbf{X}_u, \mathbf{N}\rangle = 0.$$

Similar for derivative w.r.t. v. Hence $\langle \mathbf{X}(u,v) - \mathbf{X}(u_0,v_0), \mathbf{N}\rangle \equiv$ and M is contained in a plane. (Why?)

Case 2: λ is a nonzero constant. Then

$$(\mathbf{X} + \frac{1}{\lambda}\mathbf{N})_u = \mathbf{X}_u + \frac{1}{\lambda}\mathbf{N}_u = 0.$$

Similar for derivative w.r.t. v. So $\mathbf{X} + \frac{1}{\lambda}\mathbf{N}$ is a constant vector $\mathbf{a}$, say. Then $|\mathbf{X} - \mathbf{a}| = 1/|\lambda|$. So M is contained a the sphere of radius $1/|\lambda|$ with center at $\mathbf{a}$. (Why?)

Local structure of the surface in terms of principal curvatures

Definition

Let p be a point in a regular surface patch. Then it is called

1. *Elliptic if* $\det(\mathcal{S}_p) > 0$.
2. *Hyperbolic if* $\det(\mathcal{S}_p) < 0$
3. *Parabolic if* $\det(\mathcal{S}_p) = 0$ *but* $\mathcal{S}_p \neq 0$.
4. *Planar if* $\mathcal{S}_p = 0$.

Local structure of the surface in terms of principal curvatures, cont.

Let M be a regular surface and $p \in M$. Let $\mathbf{e}_1, \mathbf{e}_2$ be the principal directions with principal curvature k_1, k_2 with $\mathbf{N} = \mathbf{e}_1 \times \mathbf{e}_2$. We choose the coordinates in $\mathbb{R}^3$ as follows: p is the origin, $\mathbf{e}_1 = (1,0,0)$, $\mathbf{e}_2 = (0,1,0)$. M is graph over xy-plane near p. That is: there is an open set $p \in V$ so that

$$M = \{(x,y,z) |\ z = f(x,y), (x,y) \in U \subset \mathbb{R}^2\}$$

where U being open in $\mathbb{R}^2$.

Local structure of the surface in terms of principal curvatures, cont.

Proposition

Near $p = (0,0,0)$, *the surface is the graph of*

$$f(x,y) = \frac{1}{2}(k_1x^2 + k_2y^2) + o(x^2 + y^2).$$

Hence locally, the regular surface patch is a

- *elliptic paraboloid if* p *is elliptic;*
- *hyperbolic paraboloid if* p *is hyperbolic;*
- *parabolic cylinder if* p *is parabolic.*

Proof

Proof: $p = (0,0,0)$ implies that $f(0,0) = 0$. $\mathbf{N} = (0,0,1)$, implies that $f_x(0,0) = 0, f_y(0,0) = 0$, we have

$$f(x,y) = \frac{1}{2}(f_{xx}(0,0)x^2 + 2f_{xy}(0,0)xy + f_{yy}(0,0)y^2) + o(x^2+y^2).$$

M can be parametrized as $\mathbf{X}(x,y) = (x, y, f(x,y))$.
Note that $\mathbf{X}_x = (1,0,f_x), \mathbf{X}_y = (0,1,f_y), \mathbf{X}_{xx} = (0,0,f_{xx}), \mathbf{X}_{xy} = \mathbf{X}_{xx} = (0,0,f_{xy}), \mathbf{X}_{yy} = (0,0,f_{yy})$.
$\mathbf{N} = (1+f_x^2+f_y^2)^{-\frac{1}{2}}(-f_x, -f_y, 1)$.

$$\mathcal{S}_p(\mathbf{e}_1) = -\frac{\partial}{\partial x}\mathbf{N} = (f_{xx}, f_{xy}, 0) = k_1\mathbf{e}_1.$$

Similar for $\mathbf{e}_2$. So at p $f_{xx} = k_1, f_{xy} = 0, f_{yy} = k_2$. Hence the result.