The second fundamental form

Definition

Let $\mathcal{S}$ be the shape operator with respect to a unit normal vector field $\mathbf{N}$, *the second fundamental form $\mathbb{II}_p$ of M at p (with respect to* $\mathbf{N}$*) is the bilinear form* $\mathbb{II}_p(\mathbf{v}, \mathbf{w}) = g(\mathcal{S}_p(\mathbf{v}), \mathbf{w}) = \langle \mathcal{S}_p(\mathbf{v}), \mathbf{w} \rangle$.

Proposition

$\mathbb{II}_p$ *is a symmetric bilinear form on* $T_p(M)$.

Proof:

$$\mathbb{II}_p(\mathbf{v}, \mathbf{w}) = \langle \mathcal{S}_p(\mathbf{v}), \mathbf{w} \rangle = \langle \mathbf{v}, \mathcal{S}_p(\mathbf{w}) \rangle = \mathbb{II}_p(\mathbf{w}, \mathbf{v})$$

because $\mathcal{S}_p$ is self-adjoint.

Coefficients of the second fundamental form

With the same notation as in the previous section of M. Let $\mathbf{N} = \mathbf{X}_u \times \mathbf{X}_v / |\mathbf{X}_u \times \mathbf{X}_v|$.

Definition

The coefficients of the second fundamental form e, f, g at p are defined as:

$$e = \mathbb{II}_p(\mathbf{X}_u, \mathbf{X}_u); f = \mathbb{II}_p(\mathbf{X}_u, \mathbf{X}_v); g = \mathbb{II}_p(\mathbf{X}_v, \mathbf{X}_v).$$

Notation: Suppose we use (u^1, u^2) as coordinates, and $\mathbf{N} = \mathbf{X}_1 \times \mathbf{X}_2 / |\mathbf{X}_1 \times \mathbf{X}_2|$, then the coefficients of the second fundamental form are denoted by

$$h_{11} = \mathbb{II}_p(\mathbf{X}_1, \mathbf{X}_1); h_{12} = \mathbb{II}_p(\mathbf{X}_1, \mathbf{X}_2) = h_{21}; h_{22} = \mathbb{II}_p(\mathbf{X}_2, \mathbf{X}_2).$$

Coefficients of the second fundamental form, cont.

$\mathcal{S}_p(\mathbf{X}_u) = -\frac{\partial}{\partial u}\mathbf{N} = -\mathbf{N}_u$. Hence

$$e = \mathbb{II}_p(\mathbf{X}_u, \mathbf{X}_u) = \langle \mathcal{S}_p(\mathbf{X}_u), \mathbf{X}_u \rangle = -\langle \mathbf{N}_u, \mathbf{X}_u \rangle = \langle \mathbf{N}, \mathbf{X}_{uu} \rangle.$$

Similarly, $f = \langle \mathbf{N}, \mathbf{X}_{uv} \rangle, g = \langle \mathbf{N}, \mathbf{X}_{vv} \rangle$.

To compute e, f, g

Proposition

$$
\begin{aligned}
e &= \langle \mathbf{N}, \mathbf{X}_{uu} \rangle = \frac{\det\left(\mathbf{X}_u, \mathbf{X}_v, \mathbf{X}_{uu}\right)}{\sqrt{EG - F^2}} \\
f &= \langle \mathbf{N}, \mathbf{X}_{uv} \rangle = \frac{\det\left(\mathbf{X}_u, \mathbf{X}_v, \mathbf{X}_{uv}\right)}{\sqrt{EG - F^2}}; \\
g &= \langle \mathbf{N}, \mathbf{X}_{vv} \rangle = \frac{\det\left(\mathbf{X}_u, \mathbf{X}_v, \mathbf{X}_{vv}\right)}{\sqrt{EG - F^2}}.
\end{aligned}
$$

Examples

Consider the torus:
$\mathbf{X}(u, v) = ((a + r\cos u)\cos v, (a + r\cos u)\sin v, r\sin u)$. Then

$$\begin{cases} \mathbf{X}_u = (-r\sin u\cos v, -r\sin u\sin v, r\cos u) \\ \mathbf{X}_v = (-(a + r\cos u)\sin v, (a + r\cos u)\cos v, 0) \\ \mathbf{X}_{uu} = (-r\cos u\cos v, -r\cos u\sin v, -r\sin u) \\ \mathbf{X}_{uv} = (r\sin u\sin v, -\sin u\cos v, 0) \\ \mathbf{X}_{vv} = (-(a + r\cos u)\cos v, -(a + r\cos u)\sin v, 0) \end{cases}$$

So $E = r^2, F = 0, G = (a + r\cos u)^2$.
$e = \det(\mathbf{X}_u, \mathbf{X}_v, \mathbf{X}_{uu})/r(a + r\cos u) = r$.
$f = 0, g = \cos u(a + r\cos u)$.

Matrix of $\mathcal{S}_p$

Recall: suppose V^2 is vector space V^2. Let $\beta = \{\mathbf{e}_1, \mathbf{e}_2\}$ be an *ordered* basis for V_2. Let $\mathbf{v} \in V^2$, then $\mathbf{v} = c_1\mathbf{e}_1 + c_2\mathbf{e}_2$. Then $[c_1, c_2]^T$ as a column vector if called the coordinates of $\mathbf{v}$ w.r.t. β, denoted by $[\mathbf{v}]_\beta$. Let T be a linear map on V^2. Then $T(\mathbf{e}_i) = \sum_{j=1}^2 a_i^j \mathbf{e}_j$. Then the matrix of T w.r.t. β is $[T]_\beta = \begin{pmatrix} a_1^1 & a_2^1 \\ a_1^2 & a_2^2 \end{pmatrix}$. We have $[T(\mathbf{v})]_\beta = [T]_\beta [\mathbf{v}]_\beta$. E.g.

$$[T(\mathbf{e}_1)]_\beta = \begin{pmatrix} a_1^1 & a_2^1 \\ a_1^2 & a_2^2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a_1^1 \\ a_1^2 \end{pmatrix}.$$

There are two invariants of T: its determinant and its trace. They are independent of the ordered basis chosen.

Gaussian curvature and mean curvature

Suppose $\mathcal{S}_p(\mathbf{X}_u) = a_1^1\mathbf{X}_u + a_1^2\mathbf{X}_v, \mathcal{S}_p(\mathbf{X}_v) = a_2^1\mathbf{X}_u + a_2^2\mathbf{X}_v$. Then the matrix of S_p with respect to the ordered basis $\beta = \{\mathbf{X}_u, \mathbf{X}_v\}$ is given by

$$[\mathcal{S}_p]_\beta = \begin{pmatrix} a_1^1 & a_2^1 \\ a_1^2 & a_2^2 \end{pmatrix}$$

Definition

The Gaussian curvature $K(p)$ of M at p is the determinant of $\mathcal{S}_p$.
The mean curvature $H(p)$ of M at p is $1/2\times$the trace of $\mathcal{S}_p$.

Gaussian curvature and mean curvature in local coordinates

Proposition

1. *The matrix of $\mathcal{S}_p$ with respect to the ordered basis $\{\mathbf{X}_u, \mathbf{X}_v\}$ is:*

$$\begin{pmatrix} a_1^1 & a_2^1 \\ a_1^2 & a_2^2 \end{pmatrix} = \begin{pmatrix} e & f \\ f & g \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}.$$

2. *The Gaussian curvature $K(p)$ and the mean curvature $H(p)$ of M at p are*

$$K(p) = \frac{eg - f^2}{EG - F^2}; H(p) = \frac{1}{2}\frac{eG - 2fF + gE}{EG - F^2}.$$

If we use coordinates (u^1, u^2) and coefficients of the first and second fundamental forms are g_{ij}, h_{ij}, then

$$K(p) = \frac{h_{11}h_{22} - h_{12}^2}{g_{11}g_{22} - g_{12}^2},$$

and

$$H(p) = \frac{1}{2}\frac{h_{11}g_{22} - 2h_{12}g_{12} + h_{22}g_{11}}{g_{11}g_{22} - g_{12}^2}.$$

Two remarks

Remark: (i) Gaussian curvature is invariant under reparametrization. (ii) Mean curvature is invariant under *orientation preserving* reparametrization.

Proof of the proposition

Proof: It is more easy to use parametrization of the form $\mathbf{X}(u^1, u^2)$. Denote $\mathbf{X}_1 = \mathbf{e}_1$, $\mathbf{X}_2 = \mathbf{e}_2$. If the matrix of $\mathcal{S}_p$ w.r.t. this ordered basis β is given above. Then $\mathcal{S}_p(\mathbf{e}_i) = \sum_{j=1}^2 a_i^j \mathbf{e}_j$. Let $g_{ij} = \langle e_i, e_j \rangle$ Now $h_{ij} = \langle \mathcal{S}_p(\mathbf{e}_i), \mathbf{e}_j \rangle = \langle \sum_k a_i^k \mathbf{e}_k, \mathbf{e}_j \rangle = \sum_k a_i^k g_{jk}$. Hence $[h_{ij}] = [\mathcal{S}]_\beta [g_{ij}]$. So

$$[\mathcal{S}]_\beta = [h_{ij}][g_{ij}]^{-1}.$$

Examples

- Let M be a plane. We know that $\mathcal{S}_p = 0$ everywhere. So the Gaussian curvature is 0, the mean curvature is zero.
- Let M be the unit sphere. If we choose $\mathbf{N}$ as before, then $\mathcal{S}$ is negative of the identity. So Gaussian curvature is 1 and mean curvature is -1.
- For the torus, and the choice of normal vector as before, we have $E = r^2, F = 0, G = (a + r\cos u)^2$.
  $e = \det(\mathbf{X}_u, \mathbf{X}_v, \mathbf{X}_{uu})/r(a + r\cos u) = r$.
  $f = 0, g = \cos u(a + r\cos u)$. Hence
  $$K = \frac{\cos u}{r(a + r\cos u)}.$$
  So $K > 0$ for $-\frac{3}{2}\pi < u < \frac{1}{2}\pi$, $K = 0$ on $u = \frac{1}{2}\pi, -\frac{3}{2}\pi$, $K < 0$ for $\frac{1}{2}\pi < u < \frac{3}{2}\pi$.