Assignment 3, Due 21/10/2022 (Friday) on or before 11:59 pm

Please upload your assignment to the Blackboard of this course

Midterm examination will be held on Oct. 18 (Tuesday), 9:30–11:15 am

(1) Let M_1, M_2 be two regular surfaces and let $F : M_1 \to M_2$ be a smooth map. Let $p \in M_1$ and $q = F(p) \in M_2$. Define the differential $dF : T_p(M_1) \to T_q(M_2)$ as follows: For any $\mathbf{v} \in T_p(M_1)$, let $\alpha(t)$ be a smooth curve on M_1 so that $\alpha(0) = p, \alpha'(0) = \mathbf{v}$. Define

$$dF_p(\mathbf{v}) = \frac{d}{dt} F(\alpha(t))|_{t=0}.$$

Prove that dF_p is well-defined and is linear.

(Consider the Gauss map $\mathbf{N} : M \to \mathbb{S}^2$. One can identify $T_p(M)$ with $T_{\mathbf{N}(p)}(\mathbb{S}^2)$. Then $\mathcal{S}_p = -d\mathbf{N}_p$.)

(2) Consider the tractrix Let $\alpha : (0, \frac{\pi}{2}) \to xz$-plane given by

$$\alpha(t) = \left(\sin\, t, 0, \cos\, t + \log\, \tan \frac{t}{2}\right).$$

Show that the Gaussian curvature of the surface of revolution obtained by rotating α about the z-axis is -1. The surface is called the pseudosphere.

(3) Let $M = \{(x, y, z)|\ z = x^2 + ky^2\}$, with $k > 0$. Show that $e_1 = (1, 0, 0)$ and $e_2 = (0, 1, 0)$ form a basis of $T_p(M)$ where $p = (0, 0, 0)$. Let $\mathbf{N}$ be the unit normal of M pointing upward, i.e. $\langle \mathbf{N}, e_3\rangle > 0$ where $e_3 = (0, 0, 1)$. Find the matrix of $\mathcal{S}_p : T_p(M) \to T_p(M)$ with respect to the ordered basis $\{e_1,\ e_2\}$. Find the principal curvatures of M at p.

(4) (Gaussian curvature of an ellipsoid) Let M be a regular orientable surface with unit normal vector field $\mathbf{N}$. Let f be a smooth function on M which is nowhere zero. Let $p \in M$ and let $\mathbf{v}_1$ and $\mathbf{v}_2$ form an orthonormal basis for $T_p(M)$.

(i) Prove that the Gaussian curvature of M at p is given by:

$$K = \frac{\langle d(f\mathbf{N})(\mathbf{v}_1) \times d(f\mathbf{N})(\mathbf{v}_2), \mathbf{N}\rangle}{f^2}.$$

Note $d(f\mathbf{N})(\mathbf{v})$ is defined as follow: let α be the curve on M with $\alpha(0) = p$, $\alpha'(0) = \mathbf{v}$, then

$$d(f\mathbf{N})(\mathbf{v}) = \frac{d}{dt}(f(\alpha(t))\mathbf{N}(\alpha(t)))\Big|_{t=0}.$$

(ii) Let M be the ellipsoid

$$h(x,y,z) := \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Let f be the restriction of the function

$$\left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right)^{\frac{1}{2}}$$

to the ellipsoid. Apply (i) to show that the Gaussian curvature of the ellipsoid is given by

$$K = \frac{1}{f^4 a^2 b^2 c^2}.$$

(See Ex. 21, p.175-176 in Do Carmo's book).

(5) The catenoid is given by

$$\mathbf{X}(u,v) = (a\cosh v\cos u, a\cosh v\sin u, av), 0 < u < 2\pi, -\infty < v < \infty,$$

$a > 0$ is a constant. Find the coefficients of the first and second fundamental forms. Show that the mean curvature of the surface is zero.