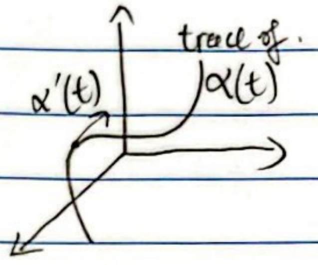
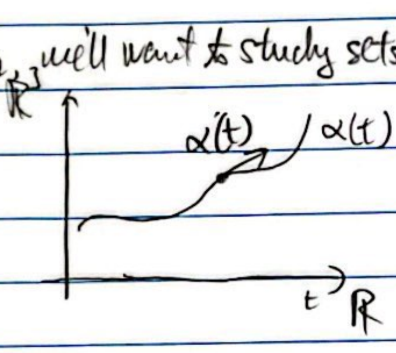
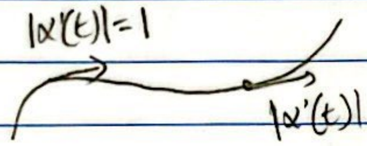
Curves in $\mathbb{R}^3$: $\alpha : I \subset \mathbb{R} \to \mathbb{R}^3$ smooth.

"sets in $\mathbb{R}^3$ that are one-dimensional" (later, we'll want to study sets in $\mathbb{R}^3$ that are "two-dim" i.e. surfaces)



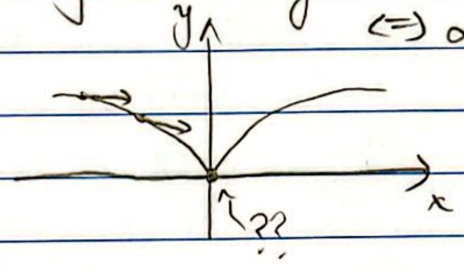


Parametrization by arc-length: curve is going at constant speed.



Can think of it as a normalization.
Often, our results won't depend on choice of parametrization, so when calculating, this is always a convenient choice.

Regularity: Essentially want a tangent vector at every point. (i.e. a 1-dim tangent space at each pt.).
$\iff \alpha'(t) \neq 0 \ \forall t \in \mathbb{R}$.



$\alpha : \mathbb{R} \to \mathbb{R}^2$ by $\alpha(t) = (t^3, t^2)$, $t \in \mathbb{R}$.

component f'n's are smooth; but still admits a singular point.

do Carmo 1-2.5: $\alpha : I \to \mathbb{R}^3$ regular curve. Show that $|\alpha(t)|$ is a nonzero constant iff $\alpha(t)$ orthogonal to $\alpha'(t)$. $\forall t \in I$.

Pf: $\Rightarrow$ Suppose $|\alpha(t)| \equiv C$ const. Then $C = |\alpha(t)| = \sqrt{\alpha(t)\cdot\alpha(t)} \iff C^2 = \alpha(t)\cdot\alpha(t)$.
$\Rightarrow \frac{d}{dt}\alpha(t)\cdot\alpha(t) = \frac{d}{dt}C^2 = 0$. But LHS $= 2\alpha'(t)\cdot\alpha(t) \Rightarrow \alpha \perp \alpha' \ \forall t$.
$\Leftarrow$ Now suppose $\alpha \perp \alpha' \ \forall t$, then $2\alpha'(t)\cdot\alpha(t) = 0 \iff \frac{d}{dt}|\alpha(t)| = 0$
$\Rightarrow |\alpha(t)|$ is constant //.

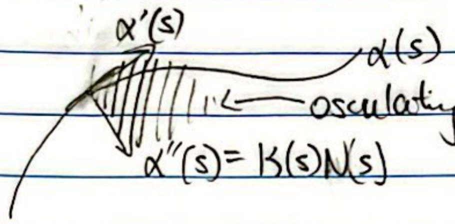
Frenet Frame:

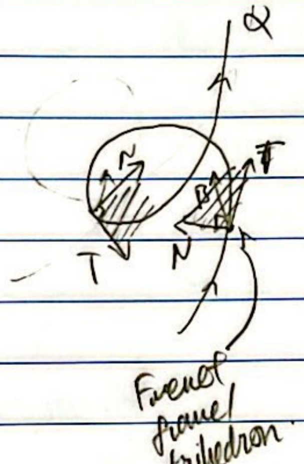
Curvature $\kappa(s) = |\alpha''(s)|$ - measures how rapidly the curve pulls away from the tangent line at s:

Normal $N(s) = \frac{1}{\kappa(s)}\alpha''(s)$. $(\kappa(s) > 0)$

Binormal $B(s) = \alpha'(s) \times N(s)$.

$\alpha'(s) \perp \alpha''(s)$ since $|\alpha'(s)| = 1 \Leftrightarrow \frac{d}{dt}|\alpha'(s)|^2 = 0 \Leftrightarrow 2\alpha'(s)\cdot\alpha''(s) = 0$.





B(s) is a unit vector normal to the osculating plane.

Torsion: $B'(s) = -\tau(s)N(s)$ (note, do Carmo has sign convention $B'(s) = \tau(s)N(s)$).

measures the degree to which the curve pulls away from the osculating plane at s.

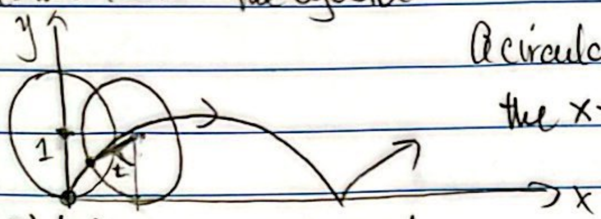
Frenet frame: $T(s) = \alpha'(s)$, $N(s) = \frac{1}{|\alpha''(s)|}\alpha''(s)$, $B(s) = T(s) \times N(s)$

all unit vectors

and the derivatives $\kappa(s) = |T'(s)|$, $B'(s) = -\tau(s)n(s)$ when expressed in the basis $\{T, N, B\}$ yields geometric information about the curve (curvature and torsion)

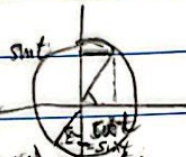
and the point is that the curvature and torsion completely determines the curve.

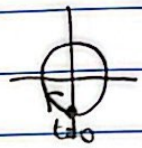
do-Carmo 1-3.2 The Cycloid.



A circular disk of radius 1 rolls without slipping along the x-axis.

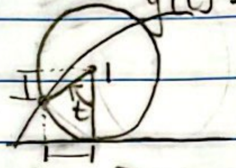
a) Obtain a parametrized curve $\alpha: \mathbb{R} \to \mathbb{R}^2$ the trace of which is the cycloid and determine its singular pts.





Determine $x(t) = t - \sin t$.

$y(t) = 1 - \cos t$.



From adding the motions of $(x,y) = (t, 1)$. with $-(\sin t, \cos t)$.

Singular Pts: $0 \stackrel{?}{=} \alpha'(t) = (1-\cos t, \sin t)$

$\sin t = 0 \iff t = k\pi, \ k \in \mathbb{Z}$.

$1 - \cos t = 0 \iff \cos t = 1 \iff t = 2k\pi, \ k \in \mathbb{Z}$.

So singular pts at $2k\pi$, $k \in \mathbb{Z}$.

b) Compute the arc-length of the cycloid corresponding to a complete rotation of the disk:

Complete rotation: $t=0$ to $t = 2\pi$.

$\alpha'(t) = (1-\cos t, \sin t)$

$|\alpha'(t)|^2 = (1-\cos t)(1-\cos t) + \sin^2 t = 1 - 2\cos t + \cos^2 t + \sin^2 t$
$= 2 - 2\cos t$.

$\Rightarrow |\alpha'(t)| = \sqrt{2 - 2\cos t}$.

$s = \int_0^{2\pi} \sqrt{2-2\cos t}\, dt$ $\qquad 1 - \cos t = 2\sin^2\left(\frac{t}{2}\right) \iff 2 - 2\cos t = 4\sin^2\left(\frac{t}{2}\right)$

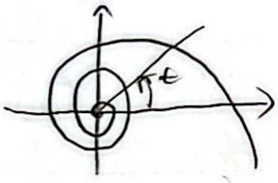
$= 2\int_0^{2\pi} \sqrt{\sin^2\left(\frac{t}{2}\right)}\, dt = 2\left[-2\cos\left(\frac{t}{2}\right)\right]_0^{2\pi} = 4(1+1) = \boxed{8}$.

Arclength.

Compute curvature and torsion of the logarithmic spiral.

$\alpha(t) = (ae^{bt}\cos t, ae^{bt}\sin t, 0)$

$a > 0, b < 0$ constants.



Torsion $= 0$ since this is a plane curve.

Arclength:

$\alpha'(t) = \left(abe^{bt}\cos t - ae^{bt}\sin t,\ abe^{bt}\sin t + ae^{bt}\cos t\right)$

$\alpha''(t) = (ab^2e^{bt}\cos t - abe^{bt}\sin t - abe^{bt}\sin t - ae^{bt}\cos t,$

$ab^2e^{bt}\sin t + abe^{bt}\cos t + abe^{bt}\cos t - ae^{bt}\sin t)$

$= (ab^2e^{bt}\cos t - 2abe^{bt}\sin t - ae^{bt}\cos t,\ ab^2e^{bt}\sin t + 2abe^{bt}\cos t - ae^{bt}\sin t)$

$|\alpha'(t)|^2 = (abe^{bt}\cos t - ae^{bt}\sin t)^2 + (abe^{bt}\sin t + ae^{bt}\cos t)^2$

$= a^2b^2e^{2bt}\cos^2 t - \cancel{2ab^2e^{2bt}\cos t\sin t} + a^2e^{2bt}\sin^2 t$

$+ a^2b^2e^{2bt}\sin^2 t + \cancel{2ab^2e^{2bt}\sin t\cos t} + a^2e^{2bt}\cos^2 t$

$= a^2b^2e^{2bt} + a^2e^{2bt} = a^2e^{2bt}(1+b^2).$

So $|\alpha'(t)| = ae^{bt}(\sqrt{1+b^2})$

$L(t_1,t_2) = \int_{t_1}^{t_2} |\alpha'(t)|\,dt = \int_{t_1}^{t_2} ae^{bt}\sqrt{1+b^2}\,dt = a\sqrt{1+b^2}\int_{t_1}^{t_2} e^{bt}\,dt = a\sqrt{1+b^2}\,\frac{1}{b}e^{bt}\Big|_{t_1}^{t_2}$

$= \frac{a}{b}\sqrt{1+b^2}\left(e^{bt_2} - e^{bt_1}\right)$

Curvature: $K(s) = \dfrac{|\alpha'(t)\times\alpha''(t)|}{|\alpha'(t)|^3}$. (use this because it is more convenient than reparametrizing by arc-length)

$$\alpha'(t)\times\alpha''(t) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ abe^{bt}\cos t - ae^{bt}\sin t & abe^{bt}\sin t + ae^{bt}\cos t & 0 \\ ab^2e^{bt}\cos t - 2abe^{bt}\sin t - ae^{bt}\cos t & ab^2e^{bt}\sin t + 2abe^{bt}\cos t - ae^{bt}\sin t & 0 \end{vmatrix}$$

$= \left[(abe^{bt}\cos t - ae^{bt}\sin t)(ab^2e^{bt}\sin t + 2abe^{bt}\cos t - ae^{bt}\sin t) - (abe^{bt}\sin t + ae^{bt}\cos t)(ab^2e^{bt}\cos t - 2abe^{bt}\sin t - ae^{bt}\cos t)\right]\hat{k}$

$= a^2(1+b^2)e^{2bt}\hat{k}.$

So $|\alpha'(t)\times\alpha''(t)| = a^2(1+b^2)e^{2bt}$, $|\alpha'(t)|^3 = a^3e^{3bt}(1+b^2)^{3/2}$

$\Rightarrow K(s) = \dfrac{1}{ae^{bt}\sqrt{1+b^2}}$. Since $b<0$, this says the curvature grows exponentially in time t.