Triangulation of surfaces

Definition

Let M be a regular surface. A triangle in M is a closed subset T which is the image of a homeomorphism $\phi : T' \to T$ where T' is a triangle in the plane $\mathbb{R}^2$. (i.e., a compact subset of $\mathbb{R}^2$ bounded by three distinct straight lines). The vertices, edges and face of T are the images of the vertices, edges and face under ϕ.

Definition

A triangulation of a compact surface M consists of a finite family of closed subsets $\{T_1, T_2, \ldots, T_n\}$ such that that any two distinct triangles, T_i and T_j are either disjoint, have a single vertex in common, or have one entire edge in common.

Facts

- Any compact surface has a triangulation.
- If M is a compact surface with boundary, then one can find a triangulation so that:

  (i) No edge has both vertices contained in the boundary unless the entire edge is contained in the boundary;
  (ii) no triangle has more than one edge contained in the boundary;
  (iii) if T_i and T_j are triangles each of which has one edge contained in the boundary, then they are either disjoint or have one vertex in common at the boundary.

Euler Characteristic

Let be a compact surface with triangulation $\{T_1, \ldots, T_n\}$. Let

$$\begin{cases} V = \text{total number of vertices;} \\ E = \text{ total number of edges;} \\ F = \text{total number of triangles (faces).} \end{cases}$$

In this case $F = n$. Then the Euler characteristic of M is defined as

$$\chi(M) = V - E + F.$$

Fact: Let $\{\mathcal{T}\}$ be a triangulation of M and let $\{\mathcal{T}'\}$ be its barycentric subdivision. Let V, E, F be the number of vertices, edges, faces of $\{\mathcal{T}\}$ and let V', E', F' be the number of vertices, edges, faces of $\{\mathcal{T}'\}$. Then

$$V - E + F = V' - E' + F'.$$

Theorem

$\chi(M)$ does not depend on the triangulation. So it is well-defined.

Classification of compact oriented surfaces without boundary

Proposition

Every compact region in a regular surface with piecewise smooth boundary has a triangulation so that each triangle is inside an isothermal coordinate neighborhood. If each triangle is positively oriented, then adjacent triangles determine opposite orientation at the common edge.

Theorem

Every oriented compact surface M without boundary is homeomoprhic to the unit sphere, or the unit sphere with g handles attached. Moreover,

$$\chi(M) = 2 - 2g$$

where g is the number of handles.

See W.S. Massey: Algebraic topology: an introduction.

The Gauss-Bonnet Theorem: global version

Theorem

Let M be an oriented regular surface and $\mathcal{R}$ is a region in M bounded by piecewise smooth simply closed curve $C_1, \ldots, C_n$ which are positively oriented. Let $\theta_1, \ldots, \theta_l$ be the set of exterior angles of $\mathcal{R}$. Then

$$\sum_{i=1}^{n} \int_{C_i} k_g ds + \iint_{\mathcal{R}} KdA + \sum_{j=1}^{l} \theta_j = 2\pi\chi(\mathcal{R}),$$

where $\chi(\mathcal{R})$ is the Euler characteristic of $\mathcal{R}$.

Proof

Proof Let $\{T_i\}_{i=1}^F$ be a triangulation in the proposition. Let ι_{ik} be the interior angles of T_i. Then

$$\sum_{i=1}^{n}\int_{C_i} k_g ds + \iint_{\mathcal{R}} KdA = \sum_{i=1}^{F}\sum_{k=1}^{3}\iota_{ik} - \pi F.$$

- E_1=number of external edges, E_2=number internal edges;
- V_1=number of external vertices, V_2=internal vertices;
- W_1=number of external vertices which are not end points of and C_i, and W_2=number external vertices which are end points of C_i.

- $W_2 = I$, i.e. the number of exterior angles.
- $3F = 2E_2 + E_1,\ V_1 = E_1.$
- $$\sum_{i=1}^{F}\sum_{k=1}^{3}\iota_{ik} = 2\pi V_2 + \pi W_1 + \sum_{j=1}^{I}(\pi - \theta_j).$$

Hence

$$\sum_{i=1}^{F}\sum_{k=1}^{3}\iota_{ik} - \pi F = 2\pi V_2 + \pi W_1 + \sum_{j=1}^{l}(\pi - \theta_j) - \pi F.$$

$$\begin{aligned}\sum_{i=1}^{F}\sum_{k=1}^{3}\iota_{ik} - \pi F + \sum_{j=1}^{l}\theta_j &= 2\pi V_2 + \pi W_1 + \pi l - \pi F \\ &= 2\pi V_2 + \pi W_1 + \pi W_2 - \pi F \\ &= 2\pi V_2 + \pi V_1 - \pi F \\ &= 2\pi V - \pi V_1 - \pi F \\ &= 2\pi(F + V) - 3\pi F - \pi V_1 \\ &= 2\pi(F + V) - 2\pi E_2 - \pi E_1 - \pi E_1 \\ &= 2\pi(V - E + F).\end{aligned}$$

Corollary

Let M be a regular orientable compact surface. Then

$$\iint_M K dA = 2\pi\chi(M) = 4\pi(1-g),$$

where g is the genus of M. Hence: (i) $\int_M K dA > 0$ if and only if M is diffeomorphic to $\mathbb{S}^2$; (ii) $\int_M K dA = 0$ if and only if M is diffeomorphic to the torus; and (iii) $\int_M K dA < 0$ if and only if M is diffeomorphic to $\mathbb{S}^2$ with g handles attached for some $g \geq 2$.

Applications

Proposition

The Euler characteristic is well-defined (for piecewise smooth triangulation).

Proposition

Let T be a geodesic triangle on a regular surface. Then

$$\iint_T K dA = -\pi + (\iota_1 + \iota_2 + \iota_3)$$

where ι_i's are the interior angle. (RHS is called the excess of the triangle T).

Digression: Beltrami equations

Theorem

Near every point in a regular surface, one can introduce an isothermal parametrization.

Explain: Let the original first fundamental form be g_{ij} with coordinates x, y. Then isothermal coordinates be u, v. We want to find $\lambda > 0$, u, v so that

$$g_{11}(x')^2 + 2g_{12}x'y' + g_{22}(y')^2 = \lambda((u')^2 + (v')^2.$$

That is:

$$\begin{pmatrix} x' & y' \end{pmatrix} \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}$$
$$= \begin{pmatrix} u' & v' \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} u' \\ v' \end{pmatrix}$$

So we need:

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} u_x & v_x \\ u_y & v_y \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$$

So

$$\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} u_x & v_x \\ u_y & v_y \end{pmatrix} = \begin{pmatrix} \frac{1}{\lambda} & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix}$$

So we have:

$$u_x(av_x + bv_y) + u_y(bv_x + cv_y) = 0.$$

And $u_x = \rho(bv_x + cv_y), u_y = -\rho(av_x + bv_y)$. Now

$$au_x^2 + 2bu_x u_y + cu_y^2 = av_x^2 + 2bv_x v_y + cv_y^2.$$

$$\rho^2(ac - b^2) = 1.$$

Hence it remains to solve the Beltrami Equation

$$\begin{cases} u_x = \frac{bv_x + cv_y}{(ac-b^2)^{\frac{1}{2}}} \\ u_y = -\frac{av_x + bv_y}{(ac-b^2)^{\frac{1}{2}}} \end{cases}$$

Let $w = u + \mathbf{i}v, z = x + \mathbf{i}y$, the equation is equivalent to

$$\frac{\partial w}{\partial \bar{z}} = \mu \frac{\partial w}{\partial z}$$

where $\mu = (c - a - 2\mathbf{i}b)/(c + a + 2\sqrt{ac - b^2})$. Note that $|\nu| < 1$.
See Courant-Hilbert: Methods of mathematical physics, vol. 2.