Simple closed curves

Definition

Let $\alpha : [a, b] \to M$ be a curve. α is said to be piecewise smooth if there exist $a = t_0 < t_1 < \cdots < t_k = b$ such that

(i) *α is continuous;*

(ii) *α is regular and is smooth on each $[t_i, t_{i+1}]$.*

- α is said to be *simple* if $\alpha(t) \neq \alpha(t')$ for $t \neq t'$.
- α is said to be *closed* if $\alpha(a) = \alpha(b)$.
- α is said to be *simple closed* if α is closed and is simple on $(a, b]$.
- α is said to be *smooth and simple closed* if α is simple closed and $\alpha(a) = \alpha(b)$ and $\alpha'(b) = \alpha'(a)$.
- α is said to be a *closed geodesic* if α is a geodesic and is smooth simple closed, so that $\alpha(b) = \alpha(a)$ and $\alpha'(b) = \alpha'(a)$.

Definition

Let M be an oriented regular surface and $\mathcal{R} \subset M$ is a bounded domain in M which is bounded by some piecewise smooth simple closed curve $\alpha_1, \ldots, \alpha_n$. Then α_i is said to be positively oriented if the unit normal $\mathbf{n} \perp \alpha'$ is such that

(i) *$\alpha', \mathbf{n}$ are positively oriented; and*

(ii) *$\mathbf{n}$ is pointing to the interior of $\mathcal{R}$.*

Facts about isothermal parametrization

Let $\mathbf{X} : U \subset \mathbb{R}^2 \to M$ be an isothermal parametrization. Namely $\langle \mathbf{X}_u, \mathbf{X}_u \rangle = \langle \mathbf{X}_v, \mathbf{X}_v \rangle = e^{2f}, \langle \mathbf{X}_u, \mathbf{X}_v \rangle = 0$.

Lemma

Let $\alpha_1(t) = \mathbf{X}(\beta_1(t))$ and $\alpha_2(t) = \mathbf{X}(\beta_2(t))$ where $\beta_i(t)$ are curves in U. Then

$$\langle \alpha_1', \alpha_2' \rangle = e^{2f} \langle \beta_1', \beta_2' \rangle.$$

at a point of intersection of α_1, α_2. (RHS is the standard inner product in $U \subset \mathbb{R}^2$.)

Proof

: Let $\beta_i = (u_i, v_i)$ for $i = 1, 2$. Then

$$\begin{aligned}\langle \alpha_1', \alpha_2' \rangle =& \langle \mathbf{X}_u u_1' + \mathbf{X}_v v_1', \langle \mathbf{X}_u u_2' + \mathbf{X}_v v_2' \rangle \\ =& e^{2f}(u_1' u_2' + v_1' v_2') = e^{2f} \langle \beta_1', \beta_2' \rangle.\end{aligned}$$

Facts about isothermal parametrization, cont.

Let
$$\mathbf{e}_1 = \frac{\mathbf{X}_u}{|\mathbf{X}_u|} = e^{-f}\mathbf{X}_u,\ \mathbf{e}_2 = \frac{\mathbf{X}_v}{|\mathbf{X}_v|} = e^{-f}\mathbf{X}_v.$$
Let $\widetilde{\mathbf{e}}_1 = (1,0), \widetilde{\mathbf{e}}_2 = (0,1)$ be the standard orhonormal basis in $U \subset \mathbb{R}^2$.

Lemma

Let $\alpha(s)$ be a regular curve on $\mathbf{X}(U) \subset M$ parametrized by arc length and let β be such that $\mathbf{X}(\beta) = \alpha$. Suppose $\theta(s)$ be smooth so that
$$\alpha'(s) = \cos\theta(s)\mathbf{e}_1 + \sin\theta(s)\mathbf{e}_2.$$
Then
$$\frac{\beta'(s)}{|\beta'(s)|} = \cos\theta(s)\widetilde{\mathbf{e}}_1 + \sin\theta(s)\widetilde{\mathbf{e}}_2.$$

Proof.

$\cos\theta(s) = \langle \alpha', \mathbf{e}_1 \rangle = \langle \frac{\beta'(s)}{|\beta'(s)|}, \widetilde{\mathbf{e}}_1 \rangle$ etc.

□

Facts about isothermal parametrization, cont.

Lemma

Let α be a piecewise smooth simple closed curve bounded a region $\mathcal{R}$ in M so that $\alpha = \mathbf{X}(\beta)$, which bounded a region in U. If α is positively oriented w.r.t. to the orientation $\mathbf{N} = \mathbf{e}_1 \times \mathbf{e}_2$, then β is positively oriented with respect to the standard orientation in U.

$\iint_{\mathcal{R}} KdA$ for $\mathcal{R}$ inside an isothermal coordinate chart

Consider the following:

- Let $\mathbf{X} : U \to M$ be an isothermal local parametrization of an oriented surface M (i.e. $E = G = e^{2f}$, $F = 0$).
- Let $\alpha = \alpha(s) = \mathbf{X}(u(s), v(s))$ $(0 \leq s \leq l)$, be a simple closed piecewise smooth curve parametrized by arc length in M so that $\beta(s) = (u(s), v(s))$ is a piecewise smooth curved in U which bounds a region D in U. Let $\mathcal{R} = \mathbf{X}(D)$. Assume α is positively oriented.
- Let L be the length of β in U and let τ be the arc length of β as a curve in $U \subset \mathbb{R}^2$. Then $\beta(s) = \beta(s(\tau))$, $0 \leq \tau \leq L$.

Lemma

In the above setting, we have:

$$\iint_{\mathcal{R}} K dA = -\int_0^L \langle \nabla_0 f, \nu_0 \rangle d\tau$$

where $\nabla_0 f = (\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v})$*. Here* τ *is the arc length of* β *as a curve in* $U \subset \mathbb{R}^2$ *and* ν_0 *is the unit outward normal of* D*.*

Recall that in the above settings, if ϕ is a function defined on M then

$$\iint_{\mathcal{R}} \phi dA = \iint_U \phi \sqrt{EG - F^2} dudv.$$

Here $\phi = \phi(u, v) = \phi(\mathbf{X}(u, v))$.

Green's theorem

In order to prove the lemma, we need the following Green's theorem (divergence theorem):

Theorem

Let Ω be a bounded domain if $\mathbb{R}^2$ and let $\gamma = \gamma(\tau)$ parametrized by arc length, $0 \leq \tau \leq l$ be the boundary curve of Ω, positively oriented. Assume that γ is piecewise smooth and connected. Let ν be the unit outward normal of Ω. Suppose P and Q are two smooth functions defined on Ω, and let $\mathbf{w} = (P, Q)$. Then

$$\iint_\Omega \left(\frac{\partial P}{\partial u} + \frac{\partial Q}{\partial v} \right) dudv = \int_0^l \langle \mathbf{w}, \nu \rangle d\tau.$$

- Note that $\operatorname{div}\mathbf{w} = \frac{\partial P}{\partial u} + \frac{\partial Q}{\partial v}$. Hence the theorem is equivalent to say:
$$\iint_\Omega \operatorname{div}\mathbf{w}\, du dv = \int_0^l \langle \mathbf{w}, \nu \rangle d\tau.$$
- The theorem is still true if the boundary consists of finitely many piecewise smooth closed curves. We have to assume that all are positively oriented.
- The theorem is still true in higher dimensions.

Sketch of proof of the Divergence Theorem for domains in $\mathbb{R}^2$

Step 1: Assume the domain D is bounded by the line segment $L : \{a \leq x \leq b, y = 0\}$ and the graph K of a function $y = \phi(x)$ over L with $\phi(x) > 0$ for $x \in (a, b)$ and $y(a) = y(b) = 0$. If $X = (0, g(x, y))$ is a smooth vector field, then

$$\begin{aligned}
\int_D \operatorname{div} X \, dxdy &= \int_D \frac{\partial g}{\partial y} dxdy \\
&= \int_a^b \left(\int_0^{\phi(x)} \frac{\partial g}{\partial y} dy \right) dx \\
&= \int_a^b \left(g(x, \phi(x)) - g(x, 0) \right) dx.
\end{aligned}$$

The outward unit normal ν of D at the boundary L is $(0, -1)$.

Hence

$$\int_L \langle X, \nu \rangle ds = -\int_a^b g(x, 0) dx$$

The unit outward normal of D at the boundary K is $(-\phi'(x), 1)/\sqrt{1 + (\phi')^2(x)}$. and

$$\int_K \langle X, \nu \rangle ds = \int_a^b \langle X, \nu \rangle \sqrt{1 + (\phi')^2(x)} dx = \int_a^b g(x, \phi(x)) dx.$$

Step 2: The theorem is true for a domain bounded by a triangle.
Step 3: The theorem is true for a domain bounded by a polygon.
Step 4: The theorem is true for a domain bounded by a piecewise smooth curve.

Proof of the lemma.

Recall that

$$K = -e^{-2f}\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right) f.$$

On the other hand,

$$EG - F^2 = e^{4f}.$$

Hence

$$\begin{aligned}\iint_{\mathcal{R}} KdA &= -\iint_U e^{-2f}\left(\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2}\right)\cdot e^{2f}\, dudv \\ &= -\iint_U \left(\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2}\right) dudv \\ &= -\int_0^L \langle \mathbf{w}, \nu_0\rangle d\tau.\end{aligned}$$

k_g in isothermal coordinates

We want to see what is the boundary integral.
Consider:

- Let $\mathbf{X}(u, v) : U \to M$ be a local isothermal parametrization of a surface M. That is: the 1st fundamental form satisfies $E = G > 0$, $F = 0$. Let $e^{2f} = E = G$.
- Let $\mathbf{e}_1 = \mathbf{X}_u / |\mathbf{X}_u| = e^{-f}\mathbf{X}_u$, and $\mathbf{e}_2 = e^{-f}\mathbf{X}_v$.
- We also assume that $\mathbf{e}_1$, $\mathbf{e}_2$ are positively oriented. That is the orientation is given by the normal of the surface $\mathbf{N} = \mathbf{X}_u \times \mathbf{X}_v / |\mathbf{X}_u \times \mathbf{X}_v|$. We want to compute the geodesic curvature of a curve w.r.t. this orientation.

Let $\alpha : [0, l]$ be a smooth regular curve on $\mathbf{X}(U)$ with arc length parametrization. Let θ_0 be an angle such that $< \alpha'(0), \mathbf{e}_1 >= \cos\theta_0$. Once we choose θ_0, then we can define a function $\theta(s)$ such that it is smooth and $\theta(0) = \theta_0$ with $\langle \alpha'(s), \mathbf{e}_1(s) \rangle = \cos\theta(s)$ and $\langle \alpha'(s), \mathbf{e}_2(s) \rangle = \sin\theta(s)$. Hence

$$\alpha'(s) = \mathbf{e}_1 \cos\theta + \mathbf{e}_2 \sin\theta.$$

Let

$$\mathbf{n} = -\mathbf{e}_1 \sin\theta + \mathbf{e}_2 \cos\theta.$$

Then α', $\mathbf{n}$ are positively oriented.

$$k_g = \frac{d\theta}{ds} + (-f_v \frac{du}{ds} + f_u \frac{dv}{ds})$$

To compute $\iint KdA + \int_0^l k_g ds$

Now go back to what we consider.

- Let $\mathbf{X} : U \to M$ be an isothermal local parametrization of an oriented surface M (i.e. $E = G = e^{2f}$, $F = 0$).
- Let $\alpha = \alpha(s) = \mathbf{X}(u(s), v(s))$ $(0 \leq s \leq l)$, be a simple closed piecewise smooth curve parametrized by arc length in M so that $\beta(s) = (u(s), v(s))$ is a piecewise smooth curved in U which bounds a region D in U. Let $\mathcal{R} = \mathbf{X}(D)$.
- Let L be the length of β in U and let τ be the arc length of β. Then $\beta(s) = \beta(s(\tau))$, $0 \leq \tau \leq L$.

Assume there exist $0 = s_0 < s_1 < \ldots, s_{k+1} = l$ so that α is continuous and smooth in each $[s_i, s_{i+1}]$. Then we have smooth functions θ on each $[s_i, s_{i+1}]$ as above.

By the previous lemma, we have

$$\begin{aligned}\iint_{\mathcal{R}} KdA &= -\int_0^L \langle \nabla_0 f, \nu_0 \rangle_0 d\tau \\ &= -\int_0^l \langle \nabla_0 f, \nu_0 \rangle_0 \frac{d\tau}{ds} ds \\ &= -\int_0^l k_g ds + \int_0^l \frac{d\theta}{ds} ds \\ &= -\int_0^l k_g ds + \sum_{i=0}^{k} (\theta(s_{i+1}) - \theta(s_i)).\end{aligned}$$

Or

$$\iint_{\mathcal{R}} K dA + \int_0^l k_g ds = \sum_{i=0}^{k} (\theta(s_{i+1}) - \theta(s_i)).$$

What is the RHS?

Jordan curve theorem

Theorem (Jordan curve theorem)

Let α be a continuous simple closed curve in $\mathbb{R}^2$ (or in $\mathbb{S}^2$), then α will separate $\mathbb{R}^2$ (or $\mathbb{S}^2$) into two components (i.e. open connected sets).

Exterior angles and interior angles

Now let $\mathcal{R} \subset M$ is a bounded domain in M which is bounded by some piecewise smooth positively oriented simple closed curve $\alpha_1, \ldots, \alpha_n$.
Denote α be one of the α_k parametrized by arc length with length ℓ. Let $0 = t_0 < t_1 < \cdots < t_{m+1} = \ell$ such that α is smooth on $[t_i, t_{i+1}]$ and α is smooth near $\alpha(0) = \alpha(\ell)$. Each $\alpha(t_i)$ $(1 \leq i \leq m)$ is called a *vertex*.

The *exterior angle* θ_i at $\alpha(t_i)$ is defined as follows. First let

$$\alpha'(t_i-) = \lim_{t<t_i, t\to t_i} \alpha'(t); \alpha'(t_i+) = \lim_{t>t_i, t\to t_i} \alpha'(t)$$

- $\alpha'(t_i-) = \alpha'(t_i+)$, then $\theta_i = 0$.
- $\alpha'(t_i-) \neq \pm\alpha'(t_i+)$. Then they are linearly independent. We define θ_i to be the *oriented angle* from $\alpha(t_i-)$ to $\alpha(t_i+)$ between $-\pi, \pi$. θ_i is positive (negative), if $\alpha(t_i-), \alpha(t_i+)$ are positively (negatively) oriented.
- $\alpha'(t_i-) = -\alpha'(t_i+)$, the $\theta_i = \pi$ or $-\pi$. The sign is determined by 'approximation'.

The *interior angle* ι_i at $\alpha(t_i)$ is defined as $\iota_i = \pi - \theta_i$.

Hopf's Umlaufsatz

Theorem (Hopf's Umlaufsatz, Theorem of Turning Tangents)

Let $\alpha : [0, l] \to \mathbb{R}^2$ be a piecewise regular, simple closed curve with $\alpha(0) = \alpha(l)$. Let $\alpha(t_1), \dots, \alpha(t_k)$, $0 = t_0 < t_1 < \cdots < t_k < t_{k+1} = l$ be the vertices of α with exterior angle θ_i. Let φ_i be smooth choice of angles defined in $[t_i, t_{i+1}]$ such that the oriented angle from the positive axis to $\alpha'(t)$ is $\varphi_i(t)$ (i.e. $\alpha' = (\cos \varphi_i(t), \sin \varphi_i(t))$) for $t \in [t_i, t_{i+1}]$. Then

$$\sum_{i=1}^{k} (\varphi_i(t_{i+1}) - \varphi_i(t_i)) + \sum_{i=0}^{k} \theta_i = \pm 2\pi.$$

It is $+1$ if α is positively oriented and -1 if it is negatively oriented, with respect to the usual orientation of $\mathbb{R}^2$.

The Gauss-Bonnet Theorem: local version

Theorem

Let $\mathbf{X} : U \to M$ *be an isothermal local parametrization of an oriented surface* M *(i.e.* $E = G = e^{2f}$, $F = 0$*). Assume that* $\mathbf{X}$ *is orientation preserving. Let* $\alpha = \alpha(s) = \mathbf{X}(u(s), v(s))$, $0 \leq s \leq l$, *be a simple closed curve parametrized by arc length so that* $(u(s), v(s))$ *bounds a region* D *in* U*. Let* $\mathcal{R} = \mathbf{X}(D)$*. Assume* α *is piecewise smooth and positively oriented. Let* $\alpha(s_0), \ldots, \alpha(s_k)$ *be the vertices of* α *with exterior angles* $\varphi_0, \ldots, \varphi_k$*, where* $0 = s_0 < s_1 < \cdots < s_k < s_{k+1} = l$*. Then*

$$\int_0^l k_g(s)ds + \iint_{\mathcal{R}} K dA + \sum_{i=0}^{k} \varphi_i = 2\pi.$$

Proof.

Since the parametrization preserves angles and orientation, by Hopf's theorem and the fact that

$$\iint_{\mathcal{R}} K dA + \int_0^l k_g ds = \sum_{i=0}^{k} (\theta(s_{i+1}) - \theta(s_i)).$$

the result follows. □

Corollary

Suppose $k = 3$, *i.e. we have a triangle then*

$$\int_0^l k_g(s)ds + \iint_{\mathcal{R}} KdA = \sum_{i=1}^{3} \iota_i - \pi,$$

where $\iota_i = \pi - \theta_i$ *are the interior angles. Hence if each side is a geodesic, then* $K > 0$ *implies the sum of the interior angles is larger than* π, *and* $K < 0$, *implies the sum of the interior angles is less than* π.