Geodesic curvature

Let M be an orientable regular surface with unit normal vector field $\mathbf{N}$. Let α be regular curve on M parametrized by arc length. Let $\alpha' = \mathbf{T}$ and let $\mathbf{n}$ be the unit vector perpendicular to $\mathbf{T}$ so that $\mathbf{T}, \mathbf{n}, \mathbf{N}$ are positively oriented. Then

$$\alpha'' = k_n \mathbf{N} + k_g \mathbf{n}.$$

k_g is called the geodesic curvature of α in M (with respect to the orientation $\mathbf{N}$).

Some basic facts

- If we consider the orientation $\widetilde{\mathbf{N}} = -\mathbf{N}$, then $\{\mathrm{T}, \widetilde{\mathbf{n}}, \widetilde{\mathbf{N}}\}$ is positively oriented if $\widetilde{\mathbf{n}} = -\mathbf{n}$. Hence the geodesic curvature $\widetilde{k}_g$ with respect to $\mathbf{N}$ is $\widetilde{k}_g = -k_g$.
- If the orientation of the curved is changed, namely, if $\beta(s) = \alpha(-s)$, say. Then the geodesic curvature of β is equal to $-k_g$ (at the same point).
- $k_g = \langle \alpha'', \mathbf{n} \rangle = \langle \alpha'', \mathbf{N} \times \alpha' \rangle = \langle \alpha' \times \alpha'', \mathbf{N} \rangle$.
- $k_g^2 + k_n^2 = \kappa^2$, where $\kappa = |\alpha''|$ is the curvature of α.

Geodesics are 'straight lines'

Definition

A regular curve on a regular surface M is called a geodesic if (i) its geodesic curvature is zero; and (ii) it is parametrized proportional to arc length.

So being geodesic means:

- $k_g = 0$ and
- $|\alpha'|$=constant.

Note that being geodesic (i.e. $k_g = 0$, with $|\alpha'|$=constant) does not depend on orientation.

Examples

In the following all curves are assumed to be parametrized by arc length.

- The geodesic curvature of a plane curve on the xy-plane is the signed curvature of the curve.
- Consider the unit sphere $\mathbb{S}^2(1)$ with center at the origin. Suppose α is a great circle. Then α'' is parallel to normal vector on the unit sphere. Hence it is a geodesic. If α is the circle with $\{z = a\} \cap \mathbb{S}^2(1)$ with $0 < a < 1$ so that $\alpha(s) = \left(b \cos \frac{s}{b}, a \sin \frac{s}{b}, a\right)$ with $b = \sqrt{1 - a^2}$. Then $k_g^2 = b^{-2} - 1$. The sign of k_g depends on the choice of the orientations of the sphere and the curve.

Examples, cont.

Consider the surface of revolution:

$$\mathbf{X}(u, v) = (f(v)\cos u, f(v)\sin u, g(v))$$

with $f_v^2 + g_v^2 = 1$ and $f > 0$. A meridian is a curve of the form $\alpha(t) = \mathbf{X}(c, t)$ for some constant c, and a parallel is a curve of the form $\alpha(t) = \mathbf{X}(t, c)$ for some constant c. Note that

$$\mathbf{X}_u = (-(f(v)\sin u, f(v)\cos u, 0), \mathbf{X}_v = (f_v \cos u, f_v \sin u, g_v).$$

For any meridian parametrized by arc length, we have

$$\alpha'' = (f'' \cos c, f'' \sin c, g'').$$

Then $\alpha'' \perp \mathbf{X}_u$ and $\alpha'' \perp \mathbf{X}_v$. Hence its geodesic curvature is zero and it is a geodesic.

Examples, cont.

If α is a parallel, then

$$\alpha'' = (-f(c)\cos u, -f(c)\sin u, 0).$$

Then $\langle \alpha'', \mathbf{X}_u \rangle = 0$ and

$$\langle \alpha'', \mathbf{X}_v \rangle = -ff_v.$$

which is zero if and only if $f_v = 0$.

Corollary

The meridians of a surface of revolution are geodesics. A parallel is a geodesic if and only if its tangent vector is parallel to the z-axis.

Examples, cont.

Proposition

Let M_1, M_2 be two oriented regular surfaces. Suppose they are tangent at a regular curve α. Then the geodesic curvatures as a curve in M_1, M_2 are the same. Here we use the same orientation along α. In particular, if α is a geodesic on M_1, then it is also a geodesic on M_2.

Geodesic curvature in local coordinates

Let M be a regular surface and $\mathbf{X}(u^1, u^2)$ be a coordinate parametrization. Let $\mathbf{N} = \mathbf{X}_1 \times \mathbf{X}_2 / |\mathbf{X}_1 \times \mathbf{X}_2|$.

Lemma

Let $\alpha(t)$ be a regular curve on M such that $\alpha(t) = \mathbf{X}(u^1(t), u^2(t))$ (t may not be proportional to arc length). Then

$$\ddot{\alpha} = \sum_{k=1}^{2} \mathbf{X}_k \left(\ddot{u}^k + \sum_{i,j=1}^{2} \Gamma_{ij}^k \dot{u}^i \dot{u}^j \right) + \mathbb{II}(\alpha', \alpha')\mathbf{N}.$$

Here $\dot{f} = \frac{df}{dt}$ etc.

Proof

Proof: We have $\mathbf{X}_{ij} = \Gamma_{ij}^k \mathbf{X}_k + h_{ij}\mathbf{N}$, where h_{ij} are the coefficients of the second fundamental from. Since $\dot{\alpha} = \sum_i \mathbf{X}_i \dot{u}^i$, we have (using summation convention)

$$\begin{aligned}
\ddot{\alpha} =& \mathbf{X}_{ij}\dot{u}^j\dot{u}^i + \mathbf{X}_i\ddot{u}^i \\
=& \dot{u}^i\dot{u}^j \left(\Gamma_{ij}^k \mathbf{X}_k + h_{ij}\mathbf{N}\right) + \mathbf{X}_i\ddot{u}^i \\
=& \sum_{k=1}^{2} \mathbf{X}_k \left(\ddot{u}^k + \sum_{i,j=1}^{2} \Gamma_{ij}^k \dot{u}^i\dot{u}^j\right) + \mathbb{II}(\dot{\alpha},\dot{\alpha})\mathbf{N}.
\end{aligned}$$

Geodesic curvature in local coordinates, cont.

Let us compute k_g, we need the following:
Lemma: Let $\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_1, \mathbf{v}_2$ be vectors in $\mathbb{R}^3$, then

$$\langle \mathbf{u}_1 \times \mathbf{u}_2, \mathbf{v}_1 \times \mathbf{v}_2 \rangle = \langle \mathbf{u}_1, \mathbf{v}_1 \rangle \langle \mathbf{u}_2, \mathbf{v}_2 \rangle - \langle \mathbf{u}_1, \mathbf{v}_2 \rangle \langle \mathbf{u}_2, \mathbf{v}_1 \rangle.$$

Proof: Let $\mathbf{e}_1, \mathbf{e}_2$ or $\mathbf{e}_3$ be the standard base vectors in $\mathbb{R}^3$.
$\mathbf{u}_1 = a_i \mathbf{e}_i$, $\mathbf{u}_2 = b_i \mathbf{e}_i$, $\mathbf{v}_1 = c_i \mathbf{e}_i$, $\mathbf{v}_2 = d_i \mathbf{e}_i$.
Then

$$\langle \mathbf{u}_1 \times \mathbf{u}_2, \mathbf{v}_1 \times \mathbf{v}_2 \rangle = a_i b_j c_k d_l \langle \mathbf{e}_i \times \mathbf{e}_j, \mathbf{e}_k \times \mathbf{e}_l \rangle.$$

$$\begin{aligned}
&\langle \mathbf{u}_1, \mathbf{v}_1 \rangle \langle \mathbf{u}_2, \mathbf{v}_2 \rangle - \langle \mathbf{u}_1, \mathbf{v}_2 \rangle \langle \mathbf{u}_2, \mathbf{v}_1 \rangle \\
&\qquad = a_i b_j c_k d_l \left(\langle \mathbf{e}_i, \mathbf{e}_k \rangle \langle \mathbf{e}_j, \mathbf{e}_l \rangle - \langle \mathbf{e}_i, \mathbf{e}_l \rangle \langle \mathbf{e}_j, \mathbf{e}_k \rangle \right)
\end{aligned}$$

Hence only need to check the relation for base vectors.

Geodesic curvature in local coordinates, cont.

Now, suppose α is parametrized by arc length, then

$$\begin{aligned} k_g =& \langle \dot\alpha \times \ddot\alpha, \mathbf{N} \rangle \\ =& \langle \dot\alpha \times \ddot\alpha, \frac{\mathbf{X}_1 \times \mathbf{X}_2}{|\mathbf{X}_1 \times \mathbf{X}_2|} \rangle \\ =& \frac{1}{\sqrt{\det(g_{ij})}} \left(\langle \dot\alpha, \mathbf{X}_1 \rangle \langle \ddot\alpha, \mathbf{X}_2 \rangle - \langle \dot\alpha, \mathbf{X}_2 \rangle \langle \ddot\alpha, \mathbf{X}_1 \rangle \right). \end{aligned}$$

To compute:

$$\begin{cases} \langle \dot\alpha, \mathbf{X}_1 \rangle = \dot u^k g_{k1} \\ \langle \ddot\alpha, \mathbf{X}_2 \rangle = \left(\ddot u^k + \sum_{i,j=1}^2 \Gamma_{ij}^k \dot u^i \dot u^j \right) g_{k2} \\ \langle \dot\alpha, \mathbf{X}_2 \rangle = \dot u^k g_{k2} \\ \langle \ddot\alpha, \mathbf{X}_1 \rangle = \left(\ddot u^k + \sum_{i,j=1}^2 \Gamma_{ij}^k \dot u^i \dot u^j \right) g_{k1}. \end{cases}$$

Geodesic curvature in local coordinates, cont.

Hence

$$k_g = \frac{1}{\sqrt{\det(g_{ij})}}\Bigg[\left(\dot{u}^k g_{k1}\right)\left(\ddot{u}^l + \sum_{i,j=1}^{2}\Gamma_{ij}^{l}\dot{u}^i\dot{u}^j\right) g_{l2} - \left(\dot{u}^k g_{k2}\right)\left(\ddot{u}^l + \sum_{i,j=1}^{2}\Gamma_{ij}^{l}\dot{u}^i\dot{u}^j\right) g_{l1}\Bigg]$$

$$= \sqrt{\det(g_{ij})}\Bigg[-\dot{u}^2\left(\ddot{u}^1 + \sum_{i,j=1}^{2}\Gamma_{ij}^{1}\dot{u}^i\dot{u}^j\right) + \dot{u}^1\left(\ddot{u}^2 + \sum_{i,j=1}^{2}\Gamma_{ij}^{2}\dot{u}^i\dot{u}^j\right)\Bigg]$$

Geodesic curvature is intrinsic

Proposition: Geodesic curvature is intrinsic. In fact, if α is parametrized by arc length, then

$$\begin{aligned} k_g =& \sqrt{\det(g_{ij})}\left[(\dot{u}^1\ddot{u}^2 - \dot{u}^2\ddot{u}^1) + (\Gamma^2_{ij}\dot{u}^1 - \Gamma^1_{ij}\dot{u}^2)\dot{u}^i\dot{u}^j\right] \\ =& \sqrt{\det(g_{ij})}\Big[\dot{u}^1\ddot{u}^2 - \dot{u}^2\ddot{u}^1 + \Gamma^2_{11}(\dot{u}^1)^3 \\ & - \Gamma^1_{22}(\dot{u}^2)^3 + \left(2\Gamma^2_{12} - \Gamma^1_{11}\right)(\dot{u}^1)^2\dot{u}^2 - \left(2\Gamma^1_{12} - \Gamma^2_{22}\right)(\dot{u}^2)^2\dot{u}^1\Big] \end{aligned}$$

Corollary: Isometry will carry geodesics to geodesics.

Equations of geodesics

Lemma: Suppose $\alpha(t)$ is a regular curve on M which satisfies

$$\ddot{u}^k + \sum_{i,j=1}^{2} \Gamma_{ij}^k \dot{u}^i \dot{u}^j = 0$$

for $k = 1, 2.$ in any coordinate chart. Then $|\alpha'|$ is constant.
Proof If α satisfies the equations, then α'' is proportional to $\mathbf{N}$. Hence $\alpha'' \perp \alpha'$ and so $\frac{d}{dt}|\alpha'|^2 = 0$.

Equations

Proposition: Let α be a regular curve on M, it is a geodesic if and only if in any local coordinates,

$$\ddot{u}^k + \sum_{i,j=1}^{2} \Gamma_{ij}^k \dot{u}^i \dot{u}^j = 0$$

for $k = 1, 2$.
That is: the acceleration on the surface is zero: $(\ddot{\alpha})^T = 0$, where $\mathbf{u}^T$ is the tangential part of $\mathbf{u}$. Or the tangent vectors are 'constant' or parallel to be precise.

Example

In polar coordinates of the xy-plane $\mathbf{X}(r, \theta) = (r\cos\theta, r\sin\theta, 0)$ we have $\Gamma_{22}^1 = -r, \Gamma_{12}^2 = r^{-1}$ and all other Γ's are zeros. So geodesic equations are:

$$\begin{cases} \ddot{r} - r(\dot{\theta})^2 = 0; \\ \ddot{\theta} + 2r^{-1}\dot{r}\dot{\theta} = 0. \end{cases}$$

Example

Consider the surface of revolution given by

$$\mathbf{X}(u,v) = (\alpha(v)\cos u, \alpha(v)\sin u, \beta(v))$$

with $\alpha > 0$. Consider $u^1 \leftrightarrow u, u^2 \leftrightarrow v$. So

$$\begin{cases} \Gamma^1_{11} = 0, \Gamma^1_{12} = \frac{\alpha'}{\alpha}, \Gamma^1_{22} = 0; \\ \Gamma^2_{11} = -\frac{\alpha\alpha'}{(\alpha')^2+(\beta')^2}, \Gamma^2_{12} = 0, \Gamma^2_{22} = \frac{\alpha'\alpha''+\beta'\beta''}{(\alpha')^2+(\beta')^2}. \end{cases}$$

Hence geodesic equations are:

$$\begin{cases} \ddot{u} + \frac{2\alpha'}{\alpha}\dot{u}\dot{v} = 0; \\ \ddot{v} - \frac{\alpha\alpha'}{(\alpha')^2+(\beta')^2}(\dot{u})^2 + \frac{\alpha'\alpha''+\beta'\beta''}{(\alpha')^2+(\beta')^2}(\dot{v})^2 = 0. \end{cases}$$

Existence of geodesic

We have the following existence of geodesic.

Proposition

At any point $p \in M$, and any vector $\mathbf{v} \in T_p(M)$, there is a geodesic $\alpha(t)$ defined on $(-\epsilon, \epsilon)$ for some $\epsilon > 0$ such that $\alpha(0) = p$ and $\alpha'(0) = \mathbf{v}$.

This follows from the following theorem on ODE.

Theorem

Let U be an open set in $\mathbb{R}^n$ and let $I_a = (-a, a) \subset \mathbb{R}$, with $a > 0$. Suppose $\mathbf{F} : U \times I_a \to \mathbb{R}^n$ is a smooth map. Then for any $\mathbf{x}_0 \in U$, there is $0 < \delta < a$, such that the following IVP has a solution:

$$\begin{cases} \mathbf{x}'(t) = & \mathbf{F}(x(t), t), \ -\delta < t < \delta; \\ \mathbf{x}(0) = & \mathbf{x}_0. \end{cases}$$

Moreover, the solutions of the IVP is unique. Namely, if $\mathbf{x}_1$ and $\mathbf{x}_2$ are two solutions of the above IVP on $(-b, b)$ for some $0 < b < a$, then $\mathbf{x}_1 = \mathbf{x}_2$.