Solution 4

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5. The mapping $D : \ell_1^1 \to \ell^1$, defined by $D(a_n) := (na_n)$, is linear and invertible, but not continuous.

   Solution. Clearly D is linear. To see that D is invertible, we show that it is 1-1 and onto.

   Suppose $D(a_n) = D(b_n)$. Then $(na_n) = (nb_n)$, that is $na_n = nb_n$ for all $n \geq 0$. Thus $a_n = b_n$ for all $n \geq 0$, and hence D is 1-1.

   For any $(y_n) \in \ell_1$, let $(x_n) := T(y_n) \in \ell_1^1$, where $T : \ell_1 \to \ell_1^1$, $T(a_n) = (a_n/n)$ is defined in Q4. Then clearly, $(y_n) = D(x_n)$. Hence D is onto.

   To see that D is not continuous, consider the sequence $\mathbf{e}_n$ which has 1 in the n-th position and zero otherwise. Then, clearly $\mathbf{e}_n/n \to 0$ as $n \to \infty$ while $D(\mathbf{e}_n/n) = \mathbf{e}_n \not\to 0$ as $n \to \infty$. Therefore D is not continuous.

   ◀

8. $T\overline{[\![A]\!]} \subseteq \overline{[\![TA]\!]}$ for a continuous linear operator T.

   Solution. Let $x \in \overline{[\![A]\!]}$. Then there is a sequence (x_n) in $[\![A]\!]$ such that $x_n \to x$. Since T is linear, it is clear that $T(x_n) \in [\![TA]\!]$. Now, it follows from the continuity of T and the definition of closure that

   $$T(x) = \lim_{n\to\infty} T(x_n) \in \overline{[\![TA]\!]}.$$

   Therefore $T\overline{[\![A]\!]} \subseteq \overline{[\![TA]\!]}$. ◀

18. If $T_n x_n \to 0$ for any choice of unit vectors x_n, then $T_n \to 0$.

   Solution. Recall that $\|T\| = \sup_{\|x\|=1} \|Tx\|$. For each n, choose a unit vector x_n such that

   $$\|T_n x_n\| \geq \|T_n\| - \frac{1}{2^n}.$$

   Since $T_n x_n \to 0$ for any choice of unit vectors x_n, we have, by letting $n \to \infty$,

   $$0 \geq \lim_{n\to\infty} \|T_n\|.$$

   Thus $\lim_{n\to\infty} \|T_n\| = 0$, which means $T_n \to 0$. ◀