Solution 2

p. 113: 8, 9, 13

8. The Weierstrass M-test (comparison test for L^∞): if $\|f_n\|_{L^\infty} \leq M_n$ where $\sum_n M_n$ converges, then $\sum_n f_n$ converges in $L^\infty(A)$ (i.e., uniformly). Use it to show that the function $f(x) := \sum_{n=1}^\infty \frac{x^n}{n^2}$ converges uniformly on $[-1, 1]$.

   Solution. Clearly,

$$\left|\frac{x^n}{n^2}\right| = \frac{|x|^n}{n^2} \leq \frac{1}{n^2} \quad \text{for any } x \in [-1, 1],$$

   and hence $\left\|\frac{x^n}{n^2}\right\|_{L^\infty} \leq \frac{1}{n^2}$. Since $\sum_{n=1}^\infty \frac{1}{n^2}$ converges, it follows from the Weierstrass M-test that $f(x) := \sum_{n=1}^\infty \frac{x^n}{n^2}$ converges uniformly on $[-1, 1]$. ◀

9. Let $f_n(x) := e^{-nx}/n$, then $\|f_n\|_{L^1[0,1]} \leq 1/n^2$, and so $\sum_n f_n$ converges in $L^1[0, 1]$.

   Solution.

$$\begin{aligned}\|f_n\|_{L^1[0,1]} &= \int_0^1 |f_n(x)|dx = \int_0^1 \frac{e^{-nx}}{n} dx \\ &= -\frac{e^{-nx}}{n^2}\Big|_0^1 \\ &= \frac{1 - e^{-n}}{n^2} \\ &\leq \frac{1}{n^2}.\end{aligned}$$

   Hence $\sum_n \|f_n\| \leq \sum_n \frac{1}{n^2} < \infty$. As mentioned in p.106 Example 7.16.1, $L^1[0, 1]$ is a Banach space. Therefore, the absolutely convergent series $\sum_n f_n$ converges in $L^1[0, 1]$. ◀

13. *Cesáro limit:* A sequence (x_n) is said to converge in the sense of Cesáro when $\frac{x_1+\cdots x_n}{n}$ converges. Show that if $a = \lim_{n\to\infty} x_n$ exists then the Cesáro limit is also a. Show that the divergent sequence $(-1)^n$ is Cesáro convergent to 0.

    Solution. Since (x_n) is convergent, it is bounded. Thus there exists $M > 0$ such that $|x_n| \leq M$ for all n.

    Let $\varepsilon > 0$. Since $a = \lim_{n\to\infty} x_n$, there exists $N_1 \in \mathbb{N}$ so that

$$|x_n - a| < \varepsilon/2 \quad \text{whenever } n \geq N_1.$$

    Choose $N_2 \in \mathbb{N}$ so large such that $N_2 \geq 2N_1(M + |a|)/\varepsilon$.

Let $s_n := \frac{x_1+\cdots x_n}{n}$. Then, for all $n \geq N := \max\{N_1, N_2\} + 1$,

$$\begin{aligned}
|s_n - a| &= \left|\frac{(x_1 - a) + (x_2 - a) + \cdots + (x_n - a)}{n}\right| \\
&\leq \left|\frac{(x_1 - a) + \cdots + (x_{N_1} - a)}{n}\right| + \left|\frac{(x_{N_1+1} - a) + \cdots + (x_n - a)}{n}\right| \\
&\leq \frac{|x_1| + |a| + \cdots + |x_{N_1}| + |a|}{n} + \frac{|x_{N_1+1} - a| + \cdots + |x_n - a|}{n} \\
&\leq \frac{N_1(M + |a|)}{n} + \frac{(n - N_1)\varepsilon/2}{n} \\
&\leq \varepsilon/2 + \varepsilon/2 \\
&= \varepsilon.
\end{aligned}$$

Hence (x_n) converges to a in the sense of Cesáro.

Clearly, $x_n := (-1)^n$ is divergent. Note that

$$s_n := \frac{x_1 + \cdots + x_n}{n} = \begin{cases} -\dfrac{1}{n} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

Hence $s_n \to 0$ as $n \to \infty$, that is $(-1)^n$ is Cesáro convergent to 0. ◄