THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 2058 Honours Mathematical Analysis I 2022-23
Tutorial 8 solutions
3rd November 2022

- Tutorial problems will be posted every Wednesday, provided there is a tutorial class on the Thursday same week. You are advised to try out the problems before attending tutorial classes, where the questions will be discussed.
- Solutions to tutorial problems will be posted after tutorial classes.
- If you have any questions, please contact Eddie Lam via echlam@math.cuhk.edu.hk or in person during office hours.

1. (a) We will show that $\lim_{x\to 1^+} \frac{x^2}{x-1} = +\infty$. Given any $M > 0$, take $\delta = \frac{1}{M} > 0$, then for x in the range $0 < x - 1 < \delta$, in particular $x > 1$, so we have

$$\frac{x^2}{x-1} \geq \frac{1}{x-1} > \frac{1}{\delta} = M.$$

(b) We will show that $\lim_{x\to 1^-} \frac{x^2}{x-1} = -\infty$, combined with part (a), this implies that $\lim_{x\to 1} \frac{x^2}{x-1}$ does not exist.
Given $M > 0$, we pick $\delta = \min\{\frac{1}{2}, \frac{1}{4M}\}$, then for x in the range of $0 < 1 - x < \delta \leq \frac{1}{2}$, we have $\frac{1}{2} < x$ and hence $x^2 \geq \frac{1}{4}$, then

$$\frac{x^2}{x-1} \leq \frac{1}{4(x-1)} < -\frac{1}{4\delta} < -M.$$

(c) We will show that $\lim_{x\to 0} \frac{\sqrt{x+1}}{x}$ does not exist by demonstrating that the one-sided limits do not agree.
To see why, let's use sequential criterion, consider the sequence $x_n = \frac{2}{n} + \frac{1}{n^2}$, notice that $\lim x_n = 0$ and $x_n > 0$ for all n. We have

$$\begin{aligned}\lim f(x_n) &= \lim \frac{\sqrt{1 + \frac{2}{n} + \frac{1}{n^2}}}{\frac{2}{n} + \frac{1}{n^2}} \\ &= \lim \frac{1 + \frac{1}{n}}{\frac{2}{n} + \frac{1}{n^2}} = \lim \frac{n^2 + n}{2n+1} > \lim \frac{n^2+n}{2n+2} = \lim \frac{n}{2} = +\infty\end{aligned}$$

Consider another sequence $y_n = -\frac{2}{n} + \frac{1}{n^2} < 0$, then again $\lim y_n = 0$, while

$$\begin{aligned}\lim f(y_n) &= \lim \frac{\sqrt{1 - \frac{2}{n} + \frac{1}{n^2}}}{-\frac{2}{n} + \frac{1}{n^2}} \\ &= \lim \frac{1 - \frac{1}{n}}{-\frac{2}{n} + \frac{1}{n^2}} = \lim \frac{n^2 - n}{-2n+1} < \lim \frac{n^2 - n}{-2n} = \lim \frac{n-1}{-2} = -\infty\end{aligned}$$

By sequential criterion, limit does not exist.

(d) We will show that $\lim_{x\to\infty} \frac{\sqrt{x+1}}{x} = 0$. Given $\epsilon > 0$, take $M = \max\{\frac{4}{\epsilon^2}, 1\} > 0$, then for $x > M$, in particular $x > 1$, so that $\sqrt{x+1} < 2\sqrt{x}$, we have

$$\frac{\sqrt{x+1}}{|x|} < \frac{2\sqrt{x}}{|x|} = \frac{2}{\sqrt{x}} < \frac{2}{\sqrt{M}} \leq \epsilon.$$

(e) We will prove that $\lim_{x\to\infty} \frac{x-\sqrt{x}}{x+\sqrt{x}} = 1$. Given any $\epsilon > 0$, take $M = \frac{4}{\epsilon^2} > 0$, then

$$\left|\frac{x-\sqrt{x}}{x+\sqrt{x}} - 1\right| = \left|\frac{2\sqrt{x}}{x+\sqrt{x}}\right| = \frac{2}{\sqrt{x}+1} < \frac{2}{\sqrt{x}} < \frac{2}{\sqrt{M}} = \epsilon.$$

2. Note that $\lim_{x\to c} f = \infty$ iff for any $M > 0$, there exists $\delta > 0$ so that $0 < |x - c| < \delta$ implies $f(x) > M$. Then for any $\epsilon > 0$, take $M = \frac{1}{\epsilon}$, then there exists $\delta > 0$ so that $0 < |x - c| < \delta$ implies $\frac{1}{f(x)} < \frac{1}{M} = \epsilon$, in other words $\lim_{x\to c} \frac{1}{f} = 0$. The reverse direction is similar.

3. If $\lim_{x\to\infty} f(x) = L$, then for any $\epsilon > 0$, there exists $M > 0$ so that $x > M$ implies that $|f(x) - L| < \epsilon$. Then for any $\epsilon > 0$, there exists $\delta = \frac{1}{M} > 0$ so that $0 < \frac{1}{x} < \frac{1}{M} = \delta$ implies that $|f(\frac{1}{x}) - L| < \epsilon$. The other direction is similar.

4. To prove that $\lim_{x\to\infty} f \circ g = L$, pick any $\epsilon > 0$, by convergence of f, there exists $M_1 > 0$ so that for $y > M_1$, we have $|f(y) - L| < \epsilon$. For this choice of M_1, by convergence of g, there exists $M_2 > 0$ so that for $x > M_2$, we have $g(x) > M_1$.

   Now combining the above, for M_2 given above, $x > M_2 \Longrightarrow g(x) > M_1 \Longrightarrow |f(g(x)) - L| < \epsilon$.

5. We will prove a more general statement: The set of discontinuity of the floor function is $\mathbb{Z}$, and for any continuous function $f : A \to \mathbb{R}$, the set of discontinuity of $\lfloor f \rfloor$ is given by

$$S = \{x \in A \cap D(A) |\, f(x) \in \mathbb{Z} \text{ and } \forall\delta > 0, \exists y \in (x-\delta, x+\delta) \cap A \text{ such that } f(y) < f(x)\}.$$

   First we note that $\lfloor x \rfloor$ is discontinuous at any $n \in \mathbb{Z}$, this is clear because for $0 < \delta < 1$, if $n < x < n + \delta$, we have $\lfloor x \rfloor = n$, and if $n - \delta < x < n$, we have $\lfloor x \rfloor = n - 1$. So the one-sided limits are different from both sides, and the function cannot be continuous.

   Now for any $r \in \mathbb{R} \setminus \mathbb{Z}$, say $n < r < n + 1$, then there exists $\delta_0 > 0$ so that $n < r - \delta_0 < r + \delta_0 < n + 1$. Then for any $\epsilon > 0$, simply take $\delta = \delta_0$, then for any $x \in (r - \delta_0, r + \delta) \subset (n, n + 1)$, we have $|\lfloor x \rfloor - \lfloor r \rfloor| = n - n = 0 < \epsilon$.

   Next we will show that for any $x \in S$, $\lfloor f \rfloor$ is discontinuous at x. Write $f(x) = N \in \mathbb{Z}$, we simply take $\epsilon = 1$, then by assumption for any $\delta > 0$, we can find $y \in (x-\delta, x+\delta) \cap A$ so that $f(y) < f(x) = N$. Therefore $|\lfloor f(y) \rfloor - \lfloor f(x) \rfloor| = 1 \geq \epsilon$.

   Now we will prove that for $x \notin S$, $\lfloor f \rfloor$ is continuous at x. If $x \in A \setminus S$, then either $f(x) \notin \mathbb{Z}$ or $f(x) \in \mathbb{Z}$ but there exists some $\delta > 0$ so that for any $y \in (x - \delta, x + \delta) \cap A$, we have $f(y) \geq f(x)$. In the first case, i.e. $f(x) \notin \mathbb{Z}$, then $N < f(x) < N + 1$ for some $N \in \mathbb{Z}$, so there is some small $\epsilon > 0$ so that $N < f(x) - \epsilon < f(x) + \epsilon < N + 1$, then by continuity of f, there is some $\delta > 0$ so that for $y \in (x - \delta, x + \delta) \cap A$, we have $f(y) \in (f(x) - \epsilon, f(x) + \epsilon) \subset (N, N + 1)$. In particular, for y in the given range, $\lfloor f(y) \rfloor = N$ is constant. So $\lfloor f \rfloor$ must be continuous at x.

In the second case, by the assumption, there exists $\delta > 0$ so that on $(x - \delta, x + \delta) \cap A$, $f(y) \geq f(x) =: N \in \mathbb{Z}$. By continuity of f at x, there is a $\delta' > 0$ so that for $y \in (x - \delta', x + \delta') \cap A$, $|f(y) - f(x)| < 1$. Then by taking $\delta'' = \min\{\delta, \delta'\}$, for y in δ''-neighborhood of x, we must have $N + 1 > f(y) \geq N$, and hence $\lfloor f(y) \rfloor = N$. Again, we have exhibited a neighborhood of x on which $\lfloor f \rfloor$ is constant, so it must be continuous at such x.

(a) For $f(x) = \sin x$, note that $S = \{n\pi : n \in \mathbb{Z}\}$.

(b) For $f(x) = 1/x$, we see that $S = \{\frac{1}{n} : n \in \mathbb{Z}\}$.

6. Let $\lim_{x \to c} f(x) = f(c) > 0$, then by taking $\epsilon = f(c) > 0$, there is a $\delta > 0$ so that for $0 < |x - c| < \delta$, we have $|f(x) - f(c)| < f(c)$. In particular, $f(c) - f(x) < f(c)$, therefore $0 < f(x)$ holds for all $|x - c| < \delta$.