THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 2058 Honours Mathematical Analysis I 2022-23
Tutorial 7
27th October 2022

- Tutorial problems will be posted every Wednesday, provided there is a tutorial class on the Thursday same week. You are advised to try out the problems before attending tutorial classes, where the questions will be discussed.
- Solutions to tutorial problems will be posted after tutorial classes.
- If you have any questions, please contact Eddie Lam via echlam@math.cuhk.edu.hk or in person during office hours.

1. (a) For any $\epsilon > 0$, take $\delta = \min\{1, 2\epsilon\} > 0$, then for any x satisfying $0 < |x-1| < \delta$, in particular $0 < x < 2$, and so $|x+1| \geq 1$, or $\frac{1}{|x+1|} \leq 1$. We have

$$\left|\frac{x}{x+1} - \frac{1}{2}\right| = \frac{|x-1|}{2|x+1|} \leq \frac{|x-1|}{2} < \frac{\delta}{2} \leq \epsilon.$$

(b) Note that we may express $(\sqrt[3]{x}-2)(\sqrt[3]{x}^2 + 2\sqrt[3]{x} + 4) = x - 8$. And also $\sqrt[3]{x}^2 + 2\sqrt[3]{x} + 4 = (\sqrt[3]{x}+1)^2 + 3 \geq 3$ for any $x \in \mathbb{R}$. For any $\epsilon > 0$, take $\delta = 3\epsilon$, then for any x satisfying $0 < |x-8| < \delta$, we have

$$|\sqrt[3]{x} - 2| = \frac{|x-8|}{\sqrt[3]{x}^2 + 2\sqrt[3]{x} + 4} < \frac{\delta}{3} = \epsilon.$$

(c) For any $\epsilon > 0$, take $\delta = \min\{1, \epsilon/7\} > 0$, then for any x satisfying $0 < |x-1| < \delta$, in particular $0 < x < 2$, and so $|x^2 + x - 1| \leq |x|^2 + |x| + 1 \leq 7$. Now we have

$$|x^3 - 2x + 1| = |x-1| \cdot |x^2 + x - 1| \leq 7|x-1| < 7\delta \leq \epsilon.$$

(d) For any $\epsilon > 0$, take $\delta = \min\{0.1, \frac{0.61}{20.81}\epsilon\}$, then for any x satisfying $0 < |x-2| < \delta$, in particular we have $1.9 < x < 2.1$. Therefore $|x^2 - 4x - 8| \leq |x|^2 + 4|x| + 8 \leq 2.1^2 + 4.1 + 8 = 20.81$, and $\frac{1}{|x^2-3|} \leq \frac{1}{1.9^2-3} = \frac{1}{0.61}$. Now we have

$$\left|\frac{x^3-2}{x^2-3} - 6\right| = \frac{|x^3 - 6x^2 + 16|}{|x^2-3|} = \frac{|x^2-4x-8|}{|x^2-3|}|x-2| < \frac{20.81}{0.61}\delta \leq \epsilon.$$

You can pick prettier numbers if you like, but it doesn't really matter.

(e) For any $\epsilon > 0$, take $\delta = \min\{1, \epsilon/2\} > 0$, then for x satisfying $0 < |x| < \delta$, we have $|x^2+1| \leq 2$, and so

$$|x\cos x(x^2+1)| = |x| \cdot |\cos x| \cdot |x^2+1| \leq 2|x| < 2\delta \leq \epsilon.$$

(f) To prove that $\lim_{x\to a} f(x) \neq L$, we will negate the $\epsilon - \delta$ definition. It suffices to demonstrate an $\epsilon > 0$ so that for any $\delta > 0$, we can find some x within the range $0 < |x-a| < \delta$ so that $|f(x) - L| \geq \epsilon$. In our case, let's pick $\epsilon = 1$, then for any $\delta > 0$, if $\delta \geq 2$, we can simply take $x = 0$, we have

$$\left|\frac{0-1}{0+1} - 1\right| = 2 > \epsilon = 1.$$

If $2 > \delta > 0$, then we may pick any $1 - \delta < x < 1$, so

$$\left|\frac{x-1}{x+1} - 1\right| = \frac{2}{|x+1|} \geq \frac{2}{2} = 1 = \epsilon.$$

(g) We will use the sequential criterion for limits of function. It suffices to demonstrate two convergent sequences (x_n) and (y_n) so that $\lim x_n = \lim y_n = 0$, with the property that $\lim(x_n + \frac{x_n}{|x_n|}) \neq \lim(y_n + \frac{y_n}{|y_n|})$. Simply take $x_n = \frac{1}{n}$ and $y_n = -\frac{1}{n}$, then $\lim x_n = \lim y_n = 0$. But $x_n + \frac{x_n}{|x_n|} = \frac{1}{n} + 1$, while $y_n + \frac{y_n}{|y_n|} = \frac{1}{n} - 1$. Clearly they converge to 1 and -1 respectively, which are not the same.

2. (a) Suppose that $\lim_{x\to c} f(x)^2 = 0$, then for any $\epsilon > 0$, there exists some $\delta > 0$ so that for $0 < |x - c| < \delta$, we have $|f(x)^2| < \epsilon^2$. Taking square root yields $|f(x)| < \epsilon$. So we are done.

   (b) Define $f(x) = \sqrt{L}$ for $x \geq c$ and $f(x) = -\sqrt{L}$ for $x < c$. Then $\lim_{x\to c} f(x) = L$ is satisfied since $f(x)^2 = L$ is a constant function. However $\lim_{x\to c} f(x)$ does not exists since the limit from both sides are different.

3. (a) For any $\epsilon > 0$, we may take $\delta = \epsilon$, so that for x in the range of $0 < |x| < \delta$, we have $|f(x)| \leq |x| < \delta = \epsilon$.

   (b) For $c \neq 0$, by density of the rational and irrational numbers, we may find a rational sequence (q_n) and an irrational sequence (r_n) so that they both converge to c, but then $\lim f(q_n) = \lim q_n = c$, whereas $\lim f(r_n) = \lim 0 = 0$, which are not the same.

4. This follows from the following claim: If (z_n) is a bounded divergent sequence, then (z_n) contains two convergent subsequences with different limits.

   Proof. By Bolzano-Weierstrass theorem, there exists a convergent subsequence (x_n) of (z_n) with limit L. Now by divergence of (z_n) itself, just fix any $\epsilon > 0$, we may construct a subsequence w_n inductively so that $|w_n - L| \geq \epsilon$ for any $n \in \mathbb{N}$. Note that (w_n) is a bounded sequence, so we may apply Bolzano-Weierstrass theorem again to obtain a convergent subsequence (y_n). Clearly $\lim y_n \neq L$ since $|y_n - L| \geq \epsilon$ for any n.

   Now given bounded function $f : \mathbb{R} \to \mathbb{R}$ so that $\lim_{x\to c} f(x)$ does not exist. We claim that there exists some (z_n) converging to c so that $(f(z_n))$ is a divergent sequence.

   *Proof.*Suppose not, i.e. for any sequence z_n converging to c, $f(z_n)$ also converges. Then for any sequences (z_n) and (z'_n) both converging to c, note that if $\lim f(z_n) \neq \lim f(z'_n)$, then we may define z''_n to be z_n for odd n and z'_n for even n, so that $\lim z''_n = c$ still but $\lim f(z''_n)$ does not exist, which contradicts the assumption. So $\lim f(z_n)$ is unique, independent on the sequence (z_n) that converges to c. By sequential criterion, it follows that $\lim_{x\to c} f(x)$ also exists, which is a contradiction.

Now pick one such (z_n) so that $\lim z_n = c$ but $(f(z_n))$ is divergent. Since f is bounded, the sequence is also bounded. Therefore, by the proposition above, we know that there are two convergent subsequences $(f(x_n))$ and $(f(y_n))$ with different limits, clearly (x_n) and (y_n) are subsequences of (z_n) so they both converge to c. This concludes the proof.