THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 2058 Honours Mathematical Analysis I 2022-23
Homework 3 solutions
5th October 2022

- Homework will be posted on both the course webpage and blackboard every Tuesday. Students are required to upload their solutions on blackboard by 23:59 p.m. next Tuesday. Additional announcement will be made if there are no homework that week.
- Please send an email to echlam@math.cuhk.edu.hk if you have any questions.

1. Let $x_n = (1 - (-1)^n + 1/n)$, consider the subsequence $y_n = x_{2n} = 1/n$ and $z_n = x_{2n+1} = 2 + 1/n$. We know that $\lim y_n = 0$ and $\lim z_n = 2$. If x_n were convergent, then by proposition 3.3, any subsequence would have the same limit. Therefore, x_n is divergent.

2. (a)

$$\begin{aligned} x_{n+1} < x_n &\iff (n+1)^{\frac{1}{n+1}} < n^{\frac{1}{n}} \\ &\iff n+1 < n^{1+\frac{1}{n}} \\ &\iff 1+\frac{1}{n} < n^{\frac{1}{n}} \\ &\iff \left(1+\frac{1}{n}\right)^n < n. \end{aligned}$$

Consider the binomial expansion $(1+1/n)^n = 1 + \frac{n}{n} + \frac{n(n-1)}{2} \cdot \frac{1}{n^2} + \dots + \frac{1}{n^{n-2}} + \frac{1}{n^n}$. Notice that the first $n-1$ terms are all smaller than or equal to 1, meanwhile the sum of the last two terms $\frac{1}{n^{n-2}} + \frac{1}{n^n}$ is also smaller than 1, therefore proving the inequality for $n > 2$.

Now $x_n = n^{\frac{1}{n}} > 1$ for any n, and $\{x_n\}$ is monotonic decreasing for $n > 2$, so by monotone convergence theorem (and disregarding the first two terms in the sequence), $x = \lim x_n$ exists.

(b) The subsequence x_{2n} also converges to the same limit x. Note that $x_{2n}^2 = (2n)^{\frac{1}{n}} = 2^{\frac{1}{n}} x_n$. Since $\lim \frac{1}{n} = 0$, we know that $\lim 2^{\frac{1}{n}} = 2^0 = 1$. More precisely, we know by Bernoulli's inequality that $0 < 2^{\frac{1}{n}} - 1 < 1 + \frac{1}{n} - 1 = \frac{1}{n}$, so by squeeze theorem we obtain the limit. Now we have

$$x^2 = \lim x_{2n}^2 = \lim 2^{\frac{1}{n}} x_n = x.$$

So x is either 1 or 0. We conclude that $x = 1$ by noting that $x_n > 1$ for all n, so the infimum cannot be 0.

3. We prove by contradiction. Suppose that $\lim x_n \neq 0$, then there is some $\epsilon > 0$ so that for all $k \in \mathbb{N}$, there is some $n_k > k$ so that $|x_{n_k}| \geq \epsilon$. Note that n_k might not be increasing with respect to k, but we may pick out an increasing sequence by inductively defining

$m_1 = n_1$, and $m_k = n_{m_{k-1}}$, then by construction $m_k > m_{k-1}$. So (x_{m_k}) defines a subsequence.

This subsequence does not admit any further subsequence converging to 0 almost immediately following from the construction, since $|x_{m_k}| \geq \epsilon$ for all k. For the $\epsilon > 0$ taken above, $|x_{m_k}| < \epsilon$ will never hold, so any subsequence would not converge to 0. This is a contradiction.