MATH 2058: Honours Mathematical Analysis I: Home Test 2
5:00 pm, 02 Dec 2022

Important Notice:

♣ The answer paper **must be submitted before 03 Dec 2022 at 5:00 pm.**
♠ The answer paper MUST BE sent to the CU Blackboard. After submitting the answer sheet, you ARE NOT Allowed to resubmit it again.
✠ **The answer paper must include your name and student ID.**

Answer ALL Questions

1. **(25 points)**
Let $C[a, b]$ be the set of all real-valued continuous functions defined on a closed and bounded interval $[a, b]$. Let $\mathcal{F} \subseteq C[a, b]$ be a non-empty subset of $C[a, b]$ that satisfies the following condition: for all elements $u, v \in \mathcal{F}$, we have $u \wedge v \in \mathcal{F}$, where $u \wedge v(x) := \min(u(x), v(x))$ for all $x \in [a, b]$.

   (i) Let $g \in C[a, b]$. Suppose that $g(x) = \inf\{h(x) : h \in \mathcal{F}; g \leq h\}$ for all $x \in [a, b]$. Show that for every $\varepsilon > 0$, there is an element $f \in \mathcal{F}$ such that $|f(x) - g(x)| < \varepsilon$ for all $x \in [a, b]$.

   (ii) Does the Part (i) hold if the function g is assumed to be bounded only?

   (iii) Does the Part (i) hold if the domain $[a, b]$ is replaced by an unbounded closed interval $[a, \infty)$?

2. **(25 points)** For $x = (x_1, ..., x_m)$ and $y = (y_1, ..., y_m)$ in $\mathbb{R}^m$, let $\|x\| := \sqrt{x_1^2 + \cdots + x_m^2}$ and recall that the inner product of x and y is defined by $\langle x, y\rangle := \sum_{k=1}^m x_k y_k$.
Let A be a matrix of order $m \times m$ and let $B := \{x \in \mathbb{R}^m : \|x\| \leq 1\}$. Define a function $q : B \longrightarrow \mathbb{R}$ by

$$q(x) := \langle Ax, x\rangle$$

for $x \in B$.

(i) Show that the set $\{\|Ax\| : x \in \mathbb{R}^m; \|x\| = 1\}$ is bounded.

(ii) Show that the function q is a Lipschitz function on B, that is, there is $C > 0$ such that $|q(x) - q(y)| \leq C\|x - y\|$ for all $x, y \in B$.

(iii) Show that

$$\sup\{\frac{|q(x) - q(y)|}{\|x - y\|} : x, y \in B; x \neq y\} = 2 \sup\{|\langle Ax, x\rangle| : x \in \mathbb{R}^m; \|x\| = 1\}.$$

$* * *$ **END OF PAPER** $* * *$