

MATH 2028 - Applications of Fubini's Theorem

GOAL: Explore more applications of Fubini's Thm and evaluate some multiple integrals

We start with some simple examples.

Example 1: Evaluate $\int_R f dV$ where

$$f: R = [0, 1] \times [1, 2] \rightarrow \mathbb{R}$$

$$f(x, y) := 1 + x^2 + xy$$

Solution: First, we note that f is cts on R , hence integrable. Thus, Fubini's Theorem applies.

$$\begin{aligned} \int_R f dV &= \int_0^1 \int_1^2 (1 + x^2 + xy) dy dx \\ &= \int_0^1 \left[y + x^2 y + \frac{1}{2} x y^2 \right]_{y=1}^{y=2} dx \\ &= \int_0^1 \left(1 + x^2 + \frac{3}{2} x \right) dx \\ &= \left[x + \frac{1}{3} x^3 + \frac{3}{4} x^2 \right]_{x=0}^{x=1} = \frac{25}{12} \end{aligned}$$

Alternatively, we can also do the iterated integral in the reversed order.

$$\begin{aligned}\int_{\mathbb{R}} f \, dV &= \int_1^2 \int_0^1 (1 + x^2 + xy) \, dx \, dy \\ &= \int_1^2 \left[x + \frac{1}{3}x^3 + \frac{1}{2}x^2y \right]_{x=0}^{x=1} dy \\ &= \int_1^2 \left(\frac{4}{3} + \frac{1}{2}y \right) dy \\ &= \left[\frac{4}{3}y + \frac{1}{4}y^2 \right]_{y=1}^{y=2} = \frac{25}{12}\end{aligned}$$

Sometimes it is easier to compute the iterated integrals in a particular order.

Example 2 : Evaluate $\int_{\mathbb{R}} f \, dV$ where

$$f: \mathbb{R} = [0, 2] \times [-1, 1] \rightarrow \mathbb{R}$$

$$f(x, y) := xy e^{x+y^2}$$

Solution: First, we note that f is cts on R , hence integrable. Thus, Fubini's Theorem applies.

$$\begin{aligned}\int_R f dV &= \int_0^2 \int_{-1}^1 x y e^{x+y^2} dy dx \\ &= \int_0^2 x e^x \left(\int_{-1}^1 y e^{y^2} dy \right) dx = 0 \\ &\quad = 0 \quad \because \text{odd function of } y\end{aligned}$$

Doing it in a different order.

$$\begin{aligned}\int_R f dV &= \int_{-1}^1 \int_0^2 x y e^{x+y^2} dx dy \\ &= \int_{-1}^1 y e^{y^2} \left(\int_0^2 x e^x dx \right) dy \\ &= \int_{-1}^1 y e^{y^2} \left[x e^x - e^x \right]_{x=0}^{x=2} dy \\ &= (e^2 + 1) \int_{-1}^1 y e^{y^2} dy = 0 \\ &\quad = 0\end{aligned}$$

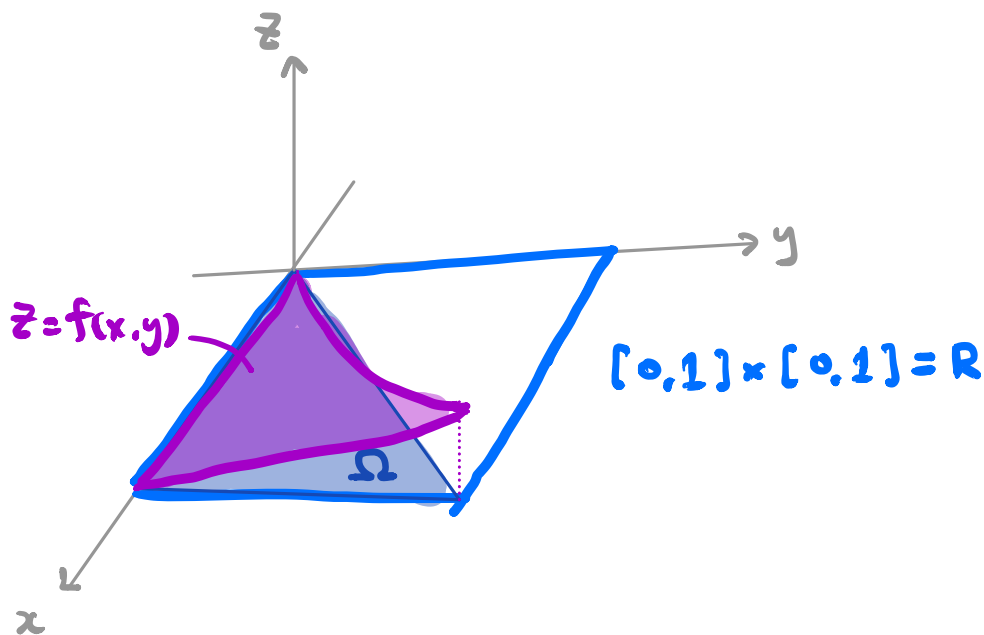
We can also use Fubini's Theorem to evaluate integrals on a **non-rectangular** bdd $\Omega \subseteq \mathbb{R}^n$.

Example 3: Find the volume of the region lying over the triangle

$$\Omega = \{ (x,y) \in [0,1] \times [0,1] \mid x \geq y \}$$

and below the graph of $f: \Omega \rightarrow \mathbb{R}$ defined by

$$f(x,y) = xy.$$



Note: Volume of the region = $\int_{\Omega} f \, dV$

Solution: Since f is cts on Ω and $\partial\Omega$ has measure zero, f is integrable on Ω and

the extension \bar{f} is integrable on R . We can apply Fubini's Theorem to evaluate $\int_R \bar{f} dV$.

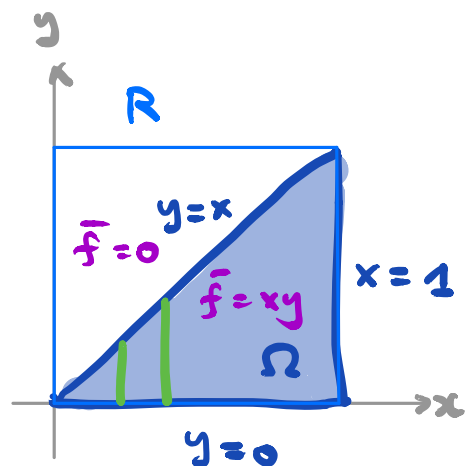
$$\int_{\Omega} f dV \stackrel{\text{by def?}}{=} \int_R \bar{f} dV \stackrel{\text{Fubini}}{=} \int_0^1 \int_0^1 \bar{f}(x,y) dy dx$$

$$= \int_0^1 \int_0^x xy dy dx$$

$$= \int_0^1 \left[\frac{1}{2} xy^2 \right]_{y=0}^{y=x} dx$$

$$= \int_0^1 \frac{1}{2} x^3 dx$$

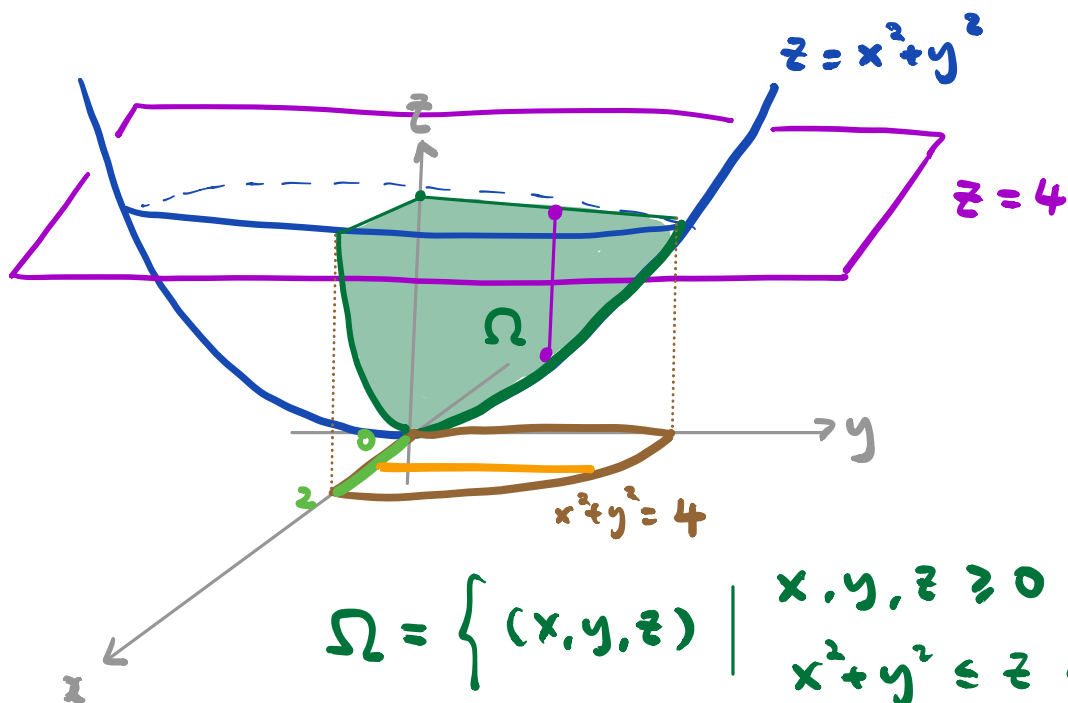
$$= \left[\frac{1}{8} x^4 \right]_{x=0}^{x=1} = \frac{1}{8}$$



Example 4: Let Ω be the region in the 1st octant bounded below by the paraboloid $z = x^2 + y^2$ and above by the plane $z = 4$. Evaluate the integral

$$\int_{\Omega} x dV$$

Solution: Step 1: Visualize the region



Since $f(x, y, z) := x$ is cts on Ω and $\partial\Omega$ has measure zero, f is integrable on Ω .

Step 2: Setup the iterated integral and evaluate.

Choose the rectangle $R = [0, 1] \times [0, 1] \times [0, 4]$

which enclose the region Ω .

$$\begin{aligned} \int_{\Omega} f \, dV &= \int_R \bar{f} \, dV \\ &= \int_0^2 \int_0^{\sqrt{4-x^2}} \int_{x^2+y^2}^4 x \, dz \, dy \, dx \end{aligned}$$

$$\begin{aligned}
&= \int_0^2 \int_0^{\sqrt{4-x^2}} x(4-x^2-y^2) dy dx \\
&= \int_0^2 x \left[(4-x^2)y - \frac{1}{3}y^3 \right]_{y=0}^{y=\sqrt{4-x^2}} dx \\
&= \int_0^2 \frac{2}{3} x (4-x^2)^{3/2} dx \\
&= \left[-\frac{2}{15} (4-x^2)^{5/2} \right]_{x=0}^{x=2} = \frac{64}{15}
\end{aligned}$$

Let's see one example in n dimension.

Example 5: Find the volume of

$$\Omega = \left\{ (x_1, \dots, x_n) : 0 \leq x_n \leq x_{n-1} \leq \dots \leq x_1 \leq 1 \right\}$$

Solution:

$$\text{Vol}(\Omega) := \int_{\Omega} 1 dV = \int_0^1 \int_0^{x_1} \dots \int_0^{x_{n-1}} dx_n \dots dx_2 dx_1$$

$$= \int_0^1 \int_0^{x_1} \dots \int_0^{x_{n-2}} x_{n-1} dx_{n-1} \dots dx_2 dx_1$$

$$= \int_0^1 \int_0^{x_1} \dots \int_0^{x_{n-3}} \frac{1}{2} x_{n-2}^2 dx_{n-2} \dots dx_2 dx_1 = \frac{1}{n!}$$

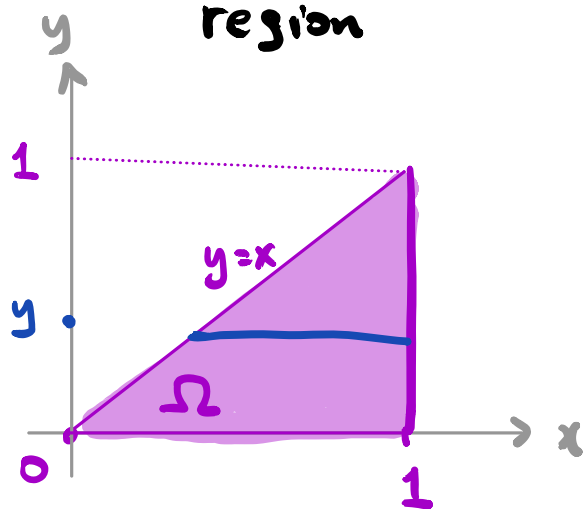
Sometimes we can turn Fubini's theorem around to help us evaluate certain iterated integrals.

Example 6: Evaluate the iterated integral

$$\int_0^1 \int_y^1 \frac{\sin x}{x} dx dy$$

Solution: Note that $\int \frac{\sin x}{x} dx$ cannot be integrated in closed form. We want to flip the order of integration to see if it helps.

Step 1: Identify the region



Step 2: Define the function

$$f(x,y) = \begin{cases} \frac{\sin x}{x} & \text{if } x > 0 \\ 1 & \text{if } x = 0 \end{cases}$$

is then a cts function on Ω .

Note: $\partial\Omega$ has measure zero.

Step 3: Apply Fubini's Theorem.

$$\int_0^1 \int_y^1 \frac{\sin x}{x} dx dy = \int_{\Omega} f dV$$

$$= \int_0^1 \int_0^x f(x,y) dy dx$$

$$= \int_0^1 \frac{\sin x}{x} \cdot x dx = 1 - \cos 1$$

Finally, we look at an example where both iterated integrals exist BUT f is NOT integrable.

Example 7: Consider $f: R = [0,1] \times [0,1] \rightarrow \mathbb{R}$

defined by

$$f(x,y) = \begin{cases} 1, & \text{if } x = \frac{m}{q}, y = \frac{n}{q} \text{ for some} \\ & m,n,q \in \mathbb{N}, q \text{ prime} \\ 0, & \text{otherwise.} \end{cases}$$

Claim: f is NOT integrable on R

Pf of Claim: Since both \mathbb{Q}^c and the set

$$\left\{ \frac{m}{q} \mid m, q \in \mathbb{N}, q \text{ prime} \right\}$$

are dense in $[0, 1]$. We have $L(f, \mathcal{P}) = 0$
and $U(f, \mathcal{P}) = 1$ for ANY partition \mathcal{P} of \mathbb{R} .
Hence, f is NOT integrable.

Next, we compute the iterated integrals.

• when $x \neq \frac{m}{q}$, $\int_0^1 f(x, y) dy = 0$

• when $x = \frac{m}{q}$, also $\int_0^1 f(x, y) dy = 0$

since $f(\frac{m}{q}, y) = 0$ except for finitely many y

Therefore, $\int_0^1 \int_0^1 f(x, y) dy dx = 0$

Similarly, we also have $\int_0^1 \int_0^1 f(x, y) dy dx = 0$

So, both iterated integrals exist and are equal to zero BUT f is NOT integrable.