Recall

Conditional distribution

- Let X, Y be discrete r.v.s. Then given $\{Y = y\}$ with $P(Y = y) > 0$,
  - the *conditional PMF*: $p_{X|Y}(x|y) := P\{X = x|Y = y\} = \dfrac{p(x, y)}{p_Y(y)}$, $\forall\, x \in \mathbb{R}$.
  - the *conditional CDF*: $F_{X|Y}(t|y) := P\{X \le t|Y = y\} = \sum_{x\le t} p_{X|Y}(x|y)$, $\forall\, t \in \mathbb{R}$.

  X, Y independent $\iff p_{X|Y}(x|y) = p_X(x)$, $\forall\, x, y \in \mathbb{R}$ with $P(Y = y) > 0$.
- Let X, Y be joint continuous r.v.s. Then for $y \in \mathbb{R}$ with $f_Y(y) > 0$,
  - the *conditional PDF*: $f_{X|Y}(x|y) := \dfrac{f_{X,Y}(x, y)}{f_Y(y)}$, $\forall\, x \in \mathbb{R}$.
  - the *conditional CDF*: $F_{X|Y}(t|y) := \int_{-\infty}^{t} f_{X|Y}(x|y)dx$, $\forall\, t \in \mathbb{R}$

  X, Y independent $\iff f_{X|Y}(x|y) = f_X(x)$, $\forall\, x, y \in \mathbb{R}$ with $f_Y(y) > 0.$,

Joint distributions of functions of random variables

Let X_1, X_2 be joint continuous random variables. For $i = 1, 2$, let $g_i \colon \mathbb{R} \to \mathbb{R}$ be some function and define $Y_i = g_i(X_1, X_2)$. Suppose

(i) for $i = 1, 2$, there exists $h_i \colon \mathbb{R} \to \mathbb{R}$ uniquely determined by $h_i(g_1(x_1), g_2(x_2)) = x_i$, $\forall\, x_i \in \mathbb{R}$.

(ii) the partial derivatives $\dfrac{\partial g_i}{\partial x_j}$, $i, j = 1, 2$ are continuous and the Jacobian $J(x_1, x_2) \neq 0$ for $x_1, x_2 \in \mathbb{R}$.

Then for $y_1, y_2 \in \mathbb{R}$,

$$\begin{aligned} f_{Y_1,Y_2}(y_1, y_2) &= f_{X_1,X_2}(x_1, x_2)\, |J(x_1, x_2)|^{-1} \\ &= f_{X_1,X_2}(h_1(y_1, y_2), h_2(y_1, y_2))\, |J\left(h_1(y_1, y_2), h_2(y_1, y_2)\right)|^{-1}. \end{aligned} \tag{1}$$

Examples

Example 1. There is a box containing 6 balls $\begin{cases} 3 & \text{blue} \\ 2 & \text{green} \\ 1 & \text{yellow.} \end{cases}$ Randomly select a ball from the box with replacement for 10 times. Let $\begin{cases} B \text{ be the number of blue balls} \\ G \text{ be the number of green balls} \\ Y \text{ be the number of yellow balls.} \end{cases}$ Find the conditional PMF of B, Y given G.

Solution. Fix any $g \in \{0, \dots, 10\}$. For any $b, y \in \{0, \dots, 10\}$ with $b + g + y = 10$,

$$\begin{aligned} p_{B,Y|G}(b, y|g) &= \frac{P\{B = b, G = g, Y = y\}}{P\{G = g\}} \\ &= \frac{\binom{10}{b,g,y}(\frac{1}{2})^b(\frac{1}{3})^g(\frac{1}{6})^y}{\binom{10}{g}(\frac{1}{3})^g(1 - \frac{1}{3})^{10-g}} \\ &= \binom{b+y}{b}(\frac{3}{4})^b(\frac{1}{4})^y. \end{aligned}$$

Hence given $\{G = g\}$ where $g = 0, \dots, 10$, B and Y are respectively the number of successes and that of failures in a Binomial experiment $\sim Bin(10 - g, 3/4)$. □

Example 2. Let X, Y be r.v.s. with joint PDF

$$f(x, y) = \begin{cases} \frac{4y}{x} & 0 < x < 1, 0 < y < x \\ 0 & \text{otherwise.} \end{cases}$$

Find the conditional PDF of Y given X and the PDF of $X + Y$.

Solution. First determine the PDF of X. For $x \in (0, 1)$,

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y)dy = \int_0^x \frac{4y}{x}dy = \frac{1}{x}\int_0^x 4ydy = 2x.$$

Then given $X = x \in (0, 1)$, for $y \in \mathbb{R}$,

$$f_{Y|X}(y|x) = \frac{f(x, y)}{f(x)} = \frac{2y}{x^2}\chi_{(0,x)}(y).$$

Next we determine the PDF of $X + Y$. (Can we use the convolution formula? No, X, Y are not independent.) Define

$$\begin{cases} U = X + Y \\ V = X \end{cases} \iff \begin{cases} X = V \\ Y = U - V \end{cases}.$$

Then (i) is satisfied. Since the Jacobian for the map $(x, y) \mapsto (x + y, x)$ is

$$J(x, y) = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = -1,$$

we have (ii) holds. Notice that $0 < x < 1, 0 < y < x$ implies $0 < u < 2, u/2 < v < u$.

Then by (1),

$$\begin{aligned} f_{U,V}(u, v) &= f(x, y)\,|J(x, y)|^{-1} \\ &= f(v, u - v) \times 1 \\ &= \begin{cases} \dfrac{4(u - v)}{v} & 0 < u < 2,\ u/2 < v < u \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Finally the PDF of $U = X + Y$ is

$$
\begin{aligned}
f_{X+Y}(u) = f_U(u) &= \int_{-\infty}^{\infty} f_{U,V}(u,v)dv \\
&= \begin{cases} \int_{u/2}^{u} \frac{4(u-v)}{v} dv & 0 < u < 1 \\ \int_{u/2}^{1} \frac{4(u-v)}{v} dv & 1 < u < 2 \end{cases} \\
&= \begin{cases} (4\ln 2 - 2)u & 0 < u < 1 \\ -4u \ln u + (4\ln 2 + 2)u - 4 & 1 < u < 2. \end{cases}
\end{aligned}
$$

Alternatively, we follow our familiar CDF-PDF-style argument. Let $Z = X + Y$. To determine the CDF of Z, since $F_Z(t) = P\{X + Y \le t\} = P\{Y \le -X + t\}$ for $t \in \mathbb{R}$, we have to compute the probability (integral) on the red shadowed region below **as t varies**. Then take differentiation to obtain the PDF.

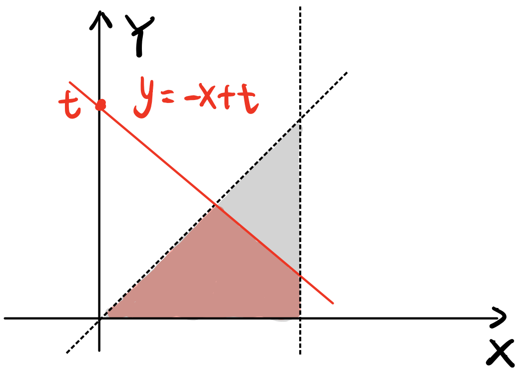

Figure 1: CDF-PDF-style

Alternatively, we can also apply the (density=mass/volume)-style argument (see e.g., [Tutorial 10, Ex. 1]). Let $Z = X + Y$. Fix any $z \in \mathbb{R}$. Let $\varepsilon > 0$ small. To determine the PDF $f_Z(z)$, we have to compute the probability (integral) on the blue shadowed region below **as z varies**. Then divide by ε and let $\varepsilon \to 0$ to obtain the PDF.

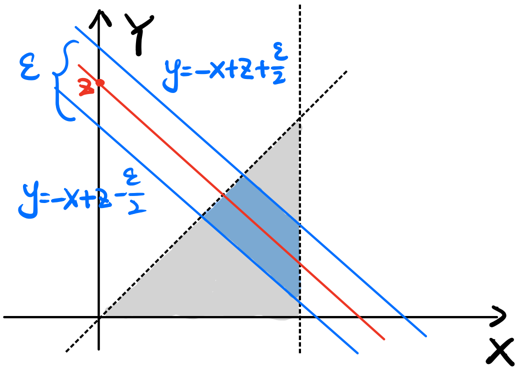

Figure 2: (density=mass/volume)-style

□

Remark. Although logically the alternative solutions in Example 2 seem more natural, the computational work might be relatively heavier.

Example 3. Let X, Y be r.v.s. with joint PDF

$$f(x,y) = \begin{cases} \dfrac{1}{x^2y^2} & x > 1, y > 1 \\ 0 & \text{otherwise.} \end{cases}$$

Find the joint PDF of $U = XY, V = X/Y$.

Solution. Since when $x > 1, y > 1$,

$$\begin{cases} u = xy \\ v = \frac{x}{y} \end{cases} \iff \begin{cases} x = \sqrt{uv} \\ y = \sqrt{\frac{u}{v}} \end{cases}$$

and

$$J(x,y) = \begin{vmatrix} y & x \\ \frac{1}{y} & -\frac{x}{y^2} \end{vmatrix} = -\frac{2x}{y},$$

the conditions for (1) are satisfied.

Notice that $x > 1, y > 1$ implies $u = xy > x/y = v, uv = x^2 > 1$. Hence for $u > v, uv > 1$,

$$f_{U,V}(u,v) = f_{X,Y}(x,y)\,|J(x,y)|^{-1} = \frac{1}{x^2y^2} \cdot \frac{y}{2x} = \frac{1}{2(uv)^{3/2}\sqrt{u/v}} = \frac{1}{2u^2v}.$$

Together we have the joint PDF of U, V

$$f_{U,V}(u,v) = \begin{cases} \frac{1}{2u^2v} & u > v, uv > 1 \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

□