During the review of questions in midterm examination, we insert the following example.

Example 1. Let $U, V \overset{i.i.d.}{\sim} U(0,1)$, i.e., U, V are independent random variables with common distribution $U(0,1)$ (the standard uniform distribution). Define

$$X \coloneqq \min(U,V) \quad \text{and} \quad Y \coloneqq \max(U,V).$$

Find a PDF f_X of X and a joint PDF $f_{X,Y}$ of X, Y.

Solution. Let F_X denote the CDF of X. Then for $t \in (0,1)$,

$$\begin{aligned} F_X(t) &= P\{X \le t\} \\ \text{(by def. of } X) \qquad &= P\{U \le t \text{ or } V \le t\} \\ \text{(De Morgan's Law)} \qquad &= 1 - P\{U > t \text{ and } V > t\} \\ \text{(by independence)} \qquad &= 1 - P\{U > t\}P\{V > t\} \\ \text{(by def. of } U, V) \qquad &= 1 - (1-t)^2 = 2t - t^2. \end{aligned}$$

Notice that $F_X(t) = 0$ if $t < 0$ and $F_X(t) = 1$ if $t > 1$. By differentiation,

$$f_X(x) = \begin{cases} 2 - 2x & x \in (0,1) \\ 0 & \text{otherwise.} \end{cases}$$

Next we will compute the joint PDF of X, Y. Fix $0 < x < y < 1$. Let $\varepsilon > 0$ be small. Then by applying the finite additivity with respect to the partition $\{U < V\} \sqcup \{U > V\} \sqcup \{U = V\}$,

$$\begin{aligned} &P\{X \in [x - \tfrac{\varepsilon}{2}, x + \tfrac{\varepsilon}{2}],\ Y \in [y - \tfrac{\varepsilon}{2}, y + \tfrac{\varepsilon}{2}]\} \\ =& P\Big(\{X \in [x - \tfrac{\varepsilon}{2}, x + \tfrac{\varepsilon}{2}],\ Y \in [y - \tfrac{\varepsilon}{2}, y + \tfrac{\varepsilon}{2}]\} \cap \{U < V\}\Big) \\ &+ P\Big(\{X \in [x - \tfrac{\varepsilon}{2}, x + \tfrac{\varepsilon}{2}],\ Y \in [y - \tfrac{\varepsilon}{2}, y + \tfrac{\varepsilon}{2}]\} \cap \{U > V\}\Big) + 0 \\ =& P\Big(U \in [x - \tfrac{\varepsilon}{2}, x + \tfrac{\varepsilon}{2}],\ V \in [y - \tfrac{\varepsilon}{2}, y + \tfrac{\varepsilon}{2}]\Big) + P\Big(V \in [x - \tfrac{\varepsilon}{2}, x + \tfrac{\varepsilon}{2}],\ U \in [y - \tfrac{\varepsilon}{2}, y + \tfrac{\varepsilon}{2}]\Big) \\ =& \varepsilon \times \varepsilon + \varepsilon \times \varepsilon \\ =& 2\varepsilon^2. \end{aligned}$$

Then

$$f_{X,Y}(x,y) = \lim_{\varepsilon \to 0} \frac{P\{X \in [x - \frac{\varepsilon}{2}, x + \frac{\varepsilon}{2}],\ Y \in [y - \frac{\varepsilon}{2}, y + \frac{\varepsilon}{2}]\}}{\varepsilon^2} = \lim_{\varepsilon \to 0} \frac{2\varepsilon^2}{\varepsilon^2} = 2. \tag{1}$$

Hence the joint PDF of X, Y is

$$f_{X,Y}(x,y) = \begin{cases} 2 & 0 < x < y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

□

Remark. In (1), we have used the local characterization of PDF at most points which can be formally remembered as (density "=" mass / volume).