Math 3280A

2022/10/07

Review

- Joint CDF of X and Y:

$$F(a,b) = P\{X \le a, Y \le b\}, \quad a, b \in \mathbb{R}.$$

- X and Y are said to be jointly cts if $\exists f: \mathbb{R}^2 \to [0,\infty)$ such that

$$P\{(X,Y) \in C\} = \iint_C f(x,y)\,dx\,dy$$

for each "measurable" set $C \subset \mathbb{R}^2$. In particular,

$$F(a,b) = \int_{-\infty}^{b}\int_{-\infty}^{a} f(x,y)\,dx\,dy.$$

- When X and Y are jointly cts with density f,

$$\frac{\partial^2 F(a,b)}{\partial a\,\partial b} = f(a,b) \quad \text{if } f \text{ is cts at } (a,b).$$

§6.2 Independent random variables.

Recall that two events E and F are said to be independent if $P(E\cap F) = P(E)P(F)$.

Def: Let X and Y be two r.v.'s.

We say that X and Y are independent if

$$P\{X\in A, Y\in B\} = P\{X\in A\}\, P\{Y\in B\},$$

for all $A, B\subset \mathbb{R}$. That is, the events $\{X\in A\}$ and $\{Y\in B\}$ are independent for all $A, B\subset \mathbb{R}$.

Remark: X and Y are independent

$$\Longleftrightarrow$$

$$F(a,b) = F_X(a)\, F_Y(b), \quad \forall\, a, b\in \mathbb{R}.$$

The direction "$\Rightarrow$" is clear. The other direction can be proved by using the three axioms of probability.

- Equivalent def of independence for r.v.'s.

<u>Lem 1</u>. Suppose X and Y are discrete. Then

X and Y are independent

$$\iff \quad p(x,y) = p_X(x)\, p_Y(y) \qquad (*)$$

Pf. Clearly X and Y are independent

$$\iff P\{X\in A, Y\in B\} = P\{X\in A\}\cdot P\{Y\in B\}.$$

Letting $A=\{x\}$, $B=\{y\}$ gives

$$p(x,y) = p_X(x)\, p_Y(y).$$

Now suppose $(*)$ holds for all x, y,

Then for given $A, B \subset \mathbb{R}$,

$$\begin{aligned} P\{X\in A, Y\in B\} &= \sum_{x\in A}\sum_{y\in B} p(x,y) \\ &= \sum_{x\in A}\sum_{y\in B} p_X(x)\, p_Y(y) \\ &= \Big(\sum_{x\in A} p_X(x)\Big)\Big(\sum_{y\in B} p_Y(y)\Big) \\ &= P\{X\in A\}\, P\{Y\in B\}. \end{aligned}$$

$\square$

Lem 2. If X and Y are jointly continuous, then X and Y are independent

$$\Leftrightarrow \quad f(x,y) = f_X(x) f_Y(y).$$

Pf. X and Y are independent

$$\Leftrightarrow \quad F(a,b) = F_X(a) F_Y(b), \quad \forall a, b \in \mathbb{R}$$

$$\Rightarrow \quad \frac{\partial^2 F(a,b)}{\partial a \partial b} = \frac{d F_X(a)}{d a} \cdot \frac{d F_Y(b)}{d b}$$

i.e. $f(a,b) = f_X(a) f_Y(b)$. (**).

Now if (**) holds, then

$$F(a,b) = \int_{-\infty}^{b} \int_{-\infty}^{a} f(x,y)\, dx\, dy$$

$$= \int_{-\infty}^{b} \int_{-\infty}^{a} f_X(x) f_Y(y)\, dx\, dy$$

$$= \left(\int_{-\infty}^{b} f_Y(y)\, dy \right) \left(\int_{-\infty}^{a} f_X(x)\, dx \right)$$

$$= F_Y(b) \cdot F_X(a).$$

Hence X, Y are independent. ∎

Example 3: Suppose X and Y have a joint density

$$f(x,y) = 24xy, \quad \text{if } 0<x<1,\ 0<y<1,\ 0<x+y<1.$$

Determine whether X and Y are independent.

<u>Solution</u>: We first calculate the marginal densities $f_X(x)$, $f_Y(y)$.

Notice that for $0<a<1$,

$$f(a,y) = \begin{cases} 24ay, & \text{if } 0<y<1-a, \\ 0 & \text{otherwise.} \end{cases}$$

So $$f_X(a) = \int_{-\infty}^{-\infty} f(a,y)\,dy$$

$$= \int_0^{1-a} 24ay\,dy$$

$$= 24a\left.\frac{y^2}{2}\right|_0^{1-a} = 12a\cdot(1-a)^2$$

Similarly,

$$f_Y(b) = \int_{-\infty}^{\infty} f(x,b)\,dx$$

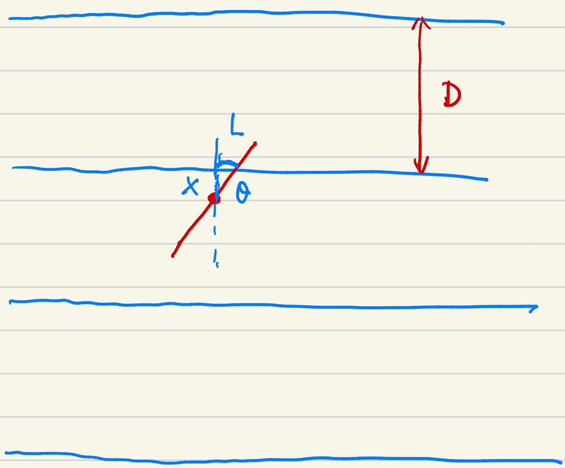
$$= \int_0^{1-b} 24\,xb\,dx$$

$$= 12b(1-b)^2 \qquad \text{if } 0<b<1.$$

Clearly $f(a,b) \neq f_X(a) f_Y(b)$. Hence X, Y are not independent. $\square$

Example 4. Buffon's needle problem.

A table is ruled with equidistant parallel lines a distance D apart. A needle of length L, where L $\leq$ D, is randomly thrown on the table. What is the probability that the needle will intersect one of the lines (the other possibility being that the needle will be completely contained in the strip between two lines)?



Solution:

Let X be the distance from the center of the needle to the nearest parallel line.

Let θ be the angle between the verticle line and the needle.

the needle intersects a parrel line

$$\Leftrightarrow \quad X \leq \tfrac{1}{2}L\cos\theta$$

We may assume that X is unif. dist on $[0, \frac{D}{2}]$

Θ is unif dist on $[0, \frac{\pi}{2}]$

and X and Θ are independent

Hence $f_X(x) = \frac{2}{D}$ for $0 < x < \frac{D}{2}$

$f_\Theta(\theta) = \frac{2}{\pi}$ if $0 < \theta < \frac{\pi}{2}$.

Now

$$P\left\{ X \leq \frac{L}{2}\cos\theta \right\}$$

$$= \iint_{\{x \leq \frac{L}{2}\cos\theta\}} f_X(x) f_\Theta(\theta)\, dx\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{L}{2}\cos\theta} \frac{4}{D\pi}\, dx\, d\theta$$

$$= \frac{L}{2D\pi}. \quad \square$$

§ 6.3 Sums of independent r.v.'s.

Question: Let X, Y be independent r.v.'s.

How to calculate the distribution of $X+Y$?

1. The Case that both X and Y are continuous.

Let X, Y have densities $f_X(x)$, $f_Y(y)$ resp.

Since X and Y are assumed to be independent,

X, Y have a joint density

$$f(x,y) = f_X(x) f_Y(y).$$

Now let $a \in \mathbb{R}$, then

$$\begin{aligned}
F_{X+Y}(a) &= P\{X+Y \leq a\} \\
&= \iint_{\substack{(x,y)\in\mathbb{R}^2: \\ x+y\leq a}} f(x,y)\,dx\,dy \\
&= \iint_{(x,y)\in\mathbb{R}^2:\ x+y\leq a} f_X(x) f_Y(y)\,dx\,dy \\
&= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{a-y} f_X(x) f_Y(y)\,dx\right)dy \\
&= \int_{-\infty}^{\infty} f_Y(y)\cdot\left(\int_{-\infty}^{a-y} f_X(x)\,dx\right)dy
\end{aligned}$$

$$= \int_{-\infty}^{\infty} f_Y(y)\, F_X(a-y)\, dy$$

$$=: F_X * f_Y(a)$$

Convolution

(For $g, h : \mathbb{R} \to \mathbb{R}$, we let

$$g * h(a) = \int_{-\infty}^{\infty} g(a-y)\, h(y)\, dy \quad)$$

$$f_{X+Y}(a) = \frac{d}{da} F_{X+Y}(a)$$

$$= \frac{d}{da} \int_{-\infty}^{\infty} F_X(a-y)\, f_Y(y)\, dy$$

$$= \int_{-\infty}^{\infty} \left(\frac{d}{da} F_X(a-y) \right) f_Y(y)\, dy$$

$$= \int_{-\infty}^{\infty} f_X(a-y)\, f_Y(y)\, dy$$

$$= f_X * f_Y(a)$$