Math 3280A 2022-10-20

Review.

- Normal r.v X with parameters μ and σ^2:

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-\mu)^2/2\sigma^2}, \quad x \in (-\infty, \infty)$$

$Z := \dfrac{X-\mu}{\sigma}$ is a standard normal r.v.

(with mean 0, Variance 1)

- Exponential r.v with parameter λ $(\lambda > 0)$,

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x > 0 \\ 0, & \text{otherwise.} \end{cases}$$

§ 5.7 The distribution of a function of a cts r.v

Q: Let X be a cts r.v. with density f_X

Let $g: \mathbb{R} \to \mathbb{R}$ be a function.

Let $Y = g(X)$.

How to find the distribution of Y?

Exer 2. Let X be a cts r.v. with density f_X.

Let $Y = X^2$.

Find the Pdf of Y.

Solution: We first calculate the CDF of Y:

$$
\begin{aligned}
F_Y(y) &= P\{Y \leq y\} \\
&= P\{X^2 \leq y\} \\
&= \begin{cases} 0 & \text{if } y < 0 \\ P\{-\sqrt{y} \leq X \leq \sqrt{y}\} & \text{if } y \geq 0 \end{cases}
\end{aligned}
$$

$$= F_X(\sqrt{y}) - F_X(-\sqrt{y})$$

Taking derivative of F_Y with respect to y gives

$$
f_Y(y) = \begin{cases} 0 & \text{if } y \leq 0 \\ f_X(\sqrt{y}) \cdot \frac{1}{2\sqrt{y}} + f_X(-\sqrt{y}) \cdot \frac{1}{2\sqrt{y}} & \text{if } y > 0 \end{cases}
$$

Ex 3. Let X be an exponential r.v with parameter λ. Let $Y = \frac{1}{X}$

Find the pdf of Y.

Solution: First notice that

$$P\{X \leq 0\} = 0$$

Now

$$F_Y(y) = P\{Y \leq y\}$$

$$= P\{\tfrac{1}{X} \leq y\}$$

$$= \begin{cases} P\{X \geq \frac{1}{y}\} & \text{if } y > 0 \\ 0 & \text{if } y < 0 \end{cases}$$

$$= \begin{cases} 1 - F_X\left(\frac{1}{y}\right) & \text{if } y > 0 \\ 0 & \text{otherwise} \end{cases}$$

Taking derivative of F_Y wrt y gives

$$f_Y(y) = \begin{cases} -f_X\left(\frac{1}{y}\right) \cdot \left(-\frac{1}{y^2}\right) & y > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} f_X\left(\frac{1}{y}\right) \cdot \frac{1}{y^2} & \text{if } y > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \lambda \cdot e^{-\lambda \frac{1}{y}} \cdot \frac{1}{y^2} & \text{if } y > 0 \\ 0 & \text{otherwise} \end{cases}$$

<u>Prop. 4</u>: Let X be a cts r.v. with density f_X. Let $g: \mathbb{R} \to \mathbb{R}$ be a differentiable, strictly monotone function. Let $Y = g(X)$, then the pdf of Y is given by

$$f_Y(y) = \begin{cases} f_X(g^{-1}(y)) \cdot \left| \dfrac{d\, g^{-1}(y)}{dy} \right| & \text{if } y = g(x) \text{ for some } x \\ 0 & \text{otherwise.} \end{cases}$$

pf. Suppose that g is strictly increasing.

Then

$$F_Y(y) = P\{ Y \le y \}$$
$$= P\{ g(X) \le y \}$$
$$= \begin{cases} 1, & \text{if } y > \max\limits_{x \in \mathbb{R}} g(x) \\ P\{ X \le g^{-1}(y) \}, & \text{if } y \in \text{Range}(g) \\ 0, & \text{if } y < \min\limits_{x \in \mathbb{R}} g(x) \end{cases}$$

$$P\{ X \le g^{-1}(y) \}$$
$$= F_X(g^{-1}(y)).$$

Taking derivative of F_Y gives

$$f_Y(y) = \begin{cases} 0 & \text{if } y \notin \text{Rang}(g) \\ f_X(g^{-1}(y)) \cdot \dfrac{d\, g^{-1}(y)}{dy}, & \text{if } y = g(x) \text{ for some } x. \end{cases}$$

Example 5. Let Z be a standard normal r.v. Find the density of Z^3.

Solution: Let $g(x) = x^3$. Then g is monotone increasing on $\mathbb{R}$ with range$(g) = \mathbb{R}$.

Hence

$$f_{Z^3}(z) = f_Z(g^{-1}(z)) \cdot \frac{d\, g^{-1}(z)}{dz}$$

$$= f_Z(z^{1/3}) \cdot \frac{1}{3} \cdot z^{-2/3}$$

$$= \frac{1}{3\sqrt{2\pi}} \cdot z^{-2/3} \cdot e^{-\frac{z^{2/3}}{2}}.$$