Math 3280A 2022-10-17

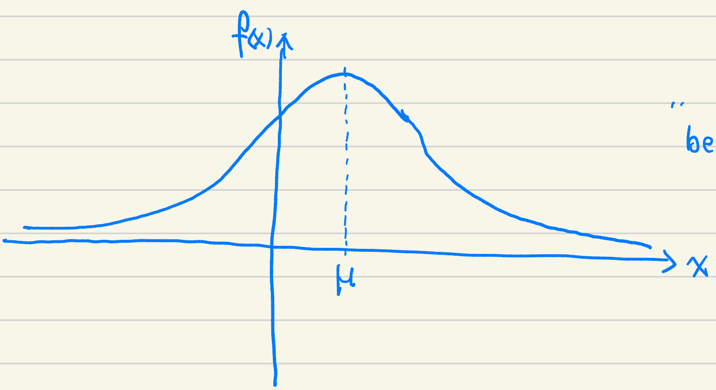
Review.

- uniform r.v. on $[a,b]$,

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{if } x \in [a,b] \\ 0 & \text{otherwise} \end{cases}$$

- Normal r.v. with parameters μ and σ^2,

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < \infty$$



"bell-shaped"

§ 5.5 Exponential r.v.

Def. Let $\lambda > 0$. Say X is an exponential r.v. with parameter λ if X has the following density

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Example: Find $E[X]$ and $V(X)$

Solution: We first estimate

$$E[X^n] \quad \text{for } n \geq 2$$

Notice that

$$E[X^n] = \int_{-\infty}^{\infty} x^n f(x)\, dx$$

$$= \int_0^{\infty} x^n \lambda e^{-\lambda x}\, dx$$

$$= \int_0^{\infty} x^n \cdot \left(-e^{-\lambda x}\right)' dx$$

$$\overset{\text{int by parts}}{=\!=} x^n \cdot \left(-e^{-\lambda x}\right)\Big|_0^{\infty} + \int_0^{\infty} e^{-\lambda x} \cdot (x^n)'\, dx$$

$$= 0 + \int_0^{\infty} n x^{n-1} e^{-\lambda x}\, dx$$

$$= 0 + \frac{n}{\lambda} \int_0^{\infty} x^{n-1} \cdot \lambda e^{-\lambda x}\, dx$$

$$= \frac{n}{\lambda} E[X^{n-1}].$$

That is,

$$(*) \quad E[X^n] = \frac{n}{\lambda} E[X^{n-1}], \quad n \geq 2.$$

Now

$$E[X] = \int_0^\infty \lambda x e^{-\lambda x} dx$$

$$= \int_0^\infty x \left(-e^{-\lambda x}\right)' dx$$

$$= x\left(-e^{-\lambda x}\right)\Big|_0^\infty + \int_0^\infty e^{-\lambda x} dx$$

$$= \frac{1}{\lambda} \int_0^\infty \lambda e^{-\lambda x} dx = \frac{1}{\lambda}$$

Using $(*)$,

$$E[X^2] = \frac{2}{\lambda} \cdot E[X]$$

$$= \frac{2}{\lambda} \cdot \frac{1}{\lambda}$$

$$= \frac{2}{\lambda^2}$$

Hence

$$V(X) = E[X^2] - \left(E[X]\right)^2$$

$$= \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}.$$

Example 1.

Suppose that the length of a phone call in minutes is an exponential random variable with parameter $\lambda = 1/10$. If someone arrives immediately ahead of you at a public telephone booth, find the probability that you will have to wait
(a) more than 10 minutes;
(b) between 10 and 20 minutes.

Solution: Let X be the waiting time in minutes.

Then X is an exp. r.v. with parameter $\lambda = \frac{1}{10}$.

Hence

(a) $$P\{X \geq 10\} = \int_{10}^{\infty} \lambda e^{-\lambda x}\, dx$$
$$= -e^{-\lambda x}\Big|_{10}^{\infty}$$
$$= e^{-\lambda \cdot 10} = e^{-1}.$$

(b) $$P\{10 \leq X \leq 20\} = -e^{-\lambda x}\Big|_{10}^{20}$$
$$= e^{-1} - e^{-2}.$$