

Review.

- Bernoulli r.v. with parameter p :

$$P\{X=1\} = p, \quad P\{X=0\} = 1-p.$$

- Binomial r.v. with parameters (n, p) :

$$P\{X=i\} = \binom{n}{i} \cdot p^i (1-p)^{n-i},$$

$$i=0, 1, \dots, n,$$

Remark: A binomial r.v. X with parameter (n, p)

can be written as

$$X = X_1 + \dots + X_n$$

where X_1, \dots, X_n are independent Bernoulli r.v.'s.

This follows from the definition of binomial r.v.

§ 4.7 Poisson r.v.

Def. Let $\lambda > 0$. A r.v. X taking values in $\{0, 1, 2, \dots\}$ is said to be a Poisson r.v. with parameter λ if

$$P\{X=i\} = e^{-\lambda} \cdot \frac{\lambda^i}{i!},$$

$i=0, 1, \dots$

Remark: $e^{\lambda} = \sum_{i=0}^{\infty} \frac{\lambda^i}{i!}.$

Hence $\sum_{i=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^i}{i!} = 1.$

A poisson r.v. can be used to approximate a binomial r.v. with parameters (n, p) when n is large, p is small so that np is of moderate size,

Let X be a binomial r.v with parameters (n, p) .

Let $np = \lambda$.

For $k=0, 1, \dots$,

$$\begin{aligned} P\{X=k\} &= \binom{n}{k} p^k \cdot (1-p)^{n-k} \\ &= \frac{n(n-1)\cdots(n-k+1)}{k!} \cdot \left(\frac{\lambda}{n}\right)^k \cdot \left(1 - \frac{\lambda}{n}\right)^{n-k} \\ &= \frac{1 \cdot \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) \cdots \left(1 - \frac{k-1}{n}\right)}{k!} \lambda^k \cdot \left(1 - \frac{\lambda}{n}\right)^n \\ &\quad \left(1 - \frac{\lambda}{n}\right)^{-k} \\ &\approx \frac{1}{k!} \lambda^k \cdot e^{-\lambda} \end{aligned}$$

□

Expected value and variance of Poisson r.v.

X — Poisson r.v. with parameter λ .

$$P\{X=k\} = e^{-\lambda} \cdot \frac{\lambda^k}{k!}, \quad k=0, 1, \dots$$

$$E[X] = \sum_{k=0}^{\infty} k \cdot e^{-\lambda} \frac{\lambda^k}{k!}$$

$$= \sum_{k=1}^{\infty} k \cdot e^{-\lambda} \cdot \frac{\lambda^k}{k!}$$

$$= \sum_{k=1}^{\infty} e^{-\lambda} \cdot \frac{\lambda^k}{(k-1)!}$$

Letting $j=k-1$

$$\sum_{j=0}^{\infty} e^{-\lambda} \cdot \lambda \cdot \frac{\lambda^j}{j!}$$

$$= \lambda \cdot \sum_{j=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^j}{j!}$$

$$= \lambda.$$

$$\begin{aligned}
E[X^2] &= \sum_{k=0}^{\infty} k^2 \cdot e^{-\lambda} \cdot \frac{\lambda^k}{k!} \\
&= \sum_{k=1}^{\infty} k \cdot e^{-\lambda} \cdot \frac{\lambda^k}{(k-1)!} \\
&= \sum_{k=1}^{\infty} (k-1) e^{-\lambda} \cdot \frac{\lambda^k}{(k-1)!} \\
&\quad + \sum_{k=1}^{\infty} e^{-\lambda} \cdot \frac{\lambda^k}{(k-1)!} \\
&= \sum_{k=2}^{\infty} e^{-\lambda} \cdot \frac{\lambda^k}{(k-2)!} + \sum_{k=1}^{\infty} e^{-\lambda} \cdot \frac{\lambda^k}{(k-1)!} \\
&= \lambda^2 + \lambda
\end{aligned}$$

So
$$\begin{aligned}
\text{Var}(X) &= E[X^2] - E[X]^2 \\
&= \lambda^2 + \lambda - \lambda^2 \\
&= \lambda.
\end{aligned}$$

§ 4.9 Expectation of sums of discrete r.v.'s.

Let X_1, X_2, \dots, X_n be discrete r.v.'s on the same sample space S .

$$\text{Prop 1. } E[X_1 + \dots + X_n] = \sum_{k=1}^n E[X_k].$$

We will prove the above result under an additional assumption that S is finite or countably infinite. (The general case is referred to Theoretical Exer 4.36 in the text book).

Lem 2. Assume that S is finite or countably infinite.

$$\text{Set } p(s) = P(\{s\}) \quad \text{for } s \in S.$$

Then for any r.v. X on S , we have

$$E[X] = \sum_{s \in S} X(s) p(s).$$

Pf. Suppose the distinct values of X
are $x_i, i \geq 1$.

$$\text{Let } S_i = \{s \in S : X(s) = x_i\}.$$

Then

$$E[X] = \sum_i x_i P\{X = x_i\}$$

$$= \sum_i x_i P(S_i)$$

$$= \sum_i x_i \sum_{s \in S_i} p(s)$$

$$= \sum_i \sum_{s \in S_i} x_i p(s)$$

$$= \sum_i \sum_{s \in S_i} X(s) p(s)$$

$$= \sum_{s \in S} X(s) p(s)$$

(because $S = \bigcup_i S_i$
with the Union being
disjoint) \square

Pf of Prop 1.

By Lem 2,

$$\begin{aligned} E[X_1 + \dots + X_n] &= \sum_{s \in S} (X_1(s) + \dots + X_n(s)) p(s) \\ &= \left(\sum_{s \in S} X_1(s) p(s) \right) + \dots + \left(\sum_{s \in S} X_n(s) p(s) \right) \\ &= E[X_1] + \dots + E[X_n]. \end{aligned}$$

\square