Math 3280A 22-09-29

Review.

Let X be a discrete r.v.
The prob. mass function of X:

$$p(a) = P\{X = a\}, \quad a \in \mathbb{R}.$$

- $$\sum_{x:\, p(x)>0} p(x) = 1.$$

Expected Value.

$$E[X] = \sum_{x:\, p(x)>0} x\, p(x).$$

Variance.

$$V(X) := E[(X-\mu)^2], \quad \text{where } \mu = E[X].$$

It describe how X is spread out from its mean.

Standard deviation: $\sqrt{V(X)}$.

- Prop. $V(X) = E[X^2] - \mu^2$, where $\mu = E[X]$.

Pf. $V(X) = E[(X-\mu)^2]$

$$= \sum_{x:\, p(x)>0} (x-\mu)^2 \cdot p(x)$$

$$= \sum_{x:\,p(x)>0} (x^2 - 2x\mu + \mu^2)\, p(x)$$

$$= \sum_{x:\,p(x)>0} x^2 p(x) - 2\mu \sum_{x:\,p(x)>0} x\, p(x) + \mu^2 \sum_{x:\,p(x)>0} p(x).$$

$$= E[X^2] - 2\mu^2 + \mu^2$$

$$= E[X^2] - \mu^2. \quad \blacksquare.$$

§ 4.6. Bernoulli r.v. and Binomial r.v.

(1) Bernoulli r.v.

Consider a random experiment, whose outcome can be classified as either a success, or a failure.

Define
$$X = \begin{cases} 1 & \text{if the outcome is a success,} \\ 0 & \text{if the outcome is a failure.} \end{cases}$$

Let $p = P\{X=1\}$, then $P\{X=0\} = 1-p$.

It has a prob. mass function: $\begin{cases} p(1) = p, \\ p(0) = 1-p. \end{cases}$

We call X a Bernoulli r.v.

- $E[X] = p$

  $E[X^2] = p$

  $V(X) = E[X^2] - E[X]^2 = p - p^2.$

(2) Binomial r.v.

Consider n independent trials, each of them results in either a success with prob p, or a failure with prob. $(1-p)$.

Let X = the number of the successes that appear in the n-trials.

We call X a Binomial r.v. with parameters (n, p).

- Example: $n=2$.

  possible outcomes $\{(S,S), (S,F), (F,S), (F,F)\}$

  $P\{(S,S)\} = P(E_1 E_2) = P(E_1)P(E_2) = p \cdot p$

  where E_1 is the event that the outcome of the first trial is S

  and E_2 is the event that the outcome of the second trial is S.

Similarly $P\{(S,F)\} = P\{(F,S)\} = p(1-p)$, $P\{(F,F)\} = (1-p)^2$.

Hence,

$$P\{X=1\} = P\{(S,F),(F,S)\} = 2\cdot p(1-p).$$

- Prob. mass function for a general Binomial r.v. with paramters (n,p).

For $i=0,1,\cdots,n$, we have

$$\boxed{P\{X=i\} = \binom{n}{i}\cdot p^i (1-p)^{n-i}.}$$

Reason: The prob. of a specual sequence of outcomes containing i successes and $(n-i)$ failures, is equal to $p^i(1-p)^{n-i}$

But there are in total $\binom{n}{i}$ such sequences, so

$$P\{X=i\} = \binom{n}{i} p^i (1-p)^{n-i}.$$

(Recall $\binom{n}{i} = \dfrac{n!}{i!\,(n-i)!} = \dfrac{n(n-1)\cdots(n-i+1)}{i(i-1)\cdots 1}$

and

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i} \quad \text{(Binomial formula)})$$

Prop. Let X be a Binomial r.v. with parameters (n,p).

Let $k\geq 1$ be an integer. Then

$$E[X^k] = np\cdot E[(Y+1)^{k-1}],$$

where Y is a Binomial r.v. with parametens $(n-1,p)$.

Pf. By def,

$$E[X^k] = \sum_{i=0}^{n} i^k \cdot \binom{n}{i} p^i (1-p)^{n-i}$$

$$= \sum_{i=1}^{n} i^k \binom{n}{i} p^i (1-p)^{n-i}$$

$$\left(\text{using } i\binom{n}{i} = n\binom{n-1}{i-1}\right)$$

$$= \sum_{i=1}^{n} n \cdot i^{k-1} \binom{n-1}{i-1} p^i (1-p)^{n-i}$$

$$= np \sum_{i=1}^{n} i^{k-1} \binom{n-1}{i-1} p^{i-1} (1-p)^{n-i}$$

Letting $j = i-1$

$$= np \sum_{j=0}^{n-1} (j+1)^{k-1} \binom{n-1}{j} p^j (1-p)^{n-1-j}$$

$$= np \cdot E[(Y+1)^{k-1}].$$

$\boxtimes$

Cor. $E[X] = np \cdot E[(Y+1)^0] = np.$

$$
\begin{aligned}
E[X^2] &= np \cdot E[(Y+1)] \\
&= np(E[Y]+1) \\
&= np((n-1)p+1)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Var}(X) &= E[X^2] - E[X]^2 \\
&= np((n-1)p+1) - (np)^2 \\
&= n(p - p^2).
\end{aligned}
$$