Math 3280A 22-09-26

Review.

- Independence of 2 or more events
- Independence of sub-experiments.
- Random variables, discrete random variables.
- Prob. mass function of a discrete r.v.

§ 4.3 Expected value.

Let X be a discrete r.v.

Let $p(x) = P\{X = x\}$ be the prob. mass function of X.

Def. The expected value of X is defined by

$$E[X] = \sum_{x:\, p(x)>0} x \cdot p(x)$$

We sometimes also call $E[X]$ the mean of X.

Hence $E[X]$ is a weighted average of the possible values of X.

Example : $X = \#\{$ heads that appear in flipping 3 fair coins $\}$

$p(0) = 1/8$, $p(1) = 3/8$, $p(2) = 3/8$, $p(3) = \frac{1}{8}$

Hence by definition,

$$E[X] = 0 \times p(0) + 1 \times p(1) + 2 \times p(2) + 3 \times p(3)$$

$$= 0 \times \frac{1}{8} + 1 \times \frac{3}{8} + 2 \times \frac{3}{8} + 3 \times \frac{1}{8}$$

$$= \frac{3+6+3}{8} = 3/2 .$$

§4.4. Expected value of a function of a r.v.

Let $X : S \to \mathbb{R}$ be a discrete r.v.

Let $g : \mathbb{R} \to \mathbb{R}$ be a function.

Then $g(X)$ is a function from S to $\mathbb{R}$,

so it is a new r.v.

Q: How can we compute $E[g(X)]$?

Remark: By def, if $y_1, y_2, \cdots,$ are the possible values of $g(X)$, then

$$E[g(X)] = \sum_j y_j P\{g(X) = y_j\}.$$

Prop. $E[g(X)] = \sum_i g(x_i) \cdot P(x_i)$,

where $x_1, x_2, \cdots,$ are all the possible values of X.

Pf. Grouping all $g(x_i)$ which take the same value, gives

$$\sum_i g(x_i)\, p(x_i)$$

$$= \sum_j \left(\sum_{i:\, g(x_i) = y_j} g(x_i)\, p(x_i) \right)$$

$$= \sum_j \left(\sum_{i:\, g(x_i) = y_j} y_j\, p(x_i) \right)$$

$$= \sum_j y_j \left(\sum_{i:\, g(x_i) = y_j} p(x_i) \right)$$

$$= \sum_j y_j\, P(g(X) = y_j)$$

This proves our desired identity. $\blacksquare$

$$\{ g(X) = y_j \} = \bigcup_{i : g(x_i) = y_j} \{ X = x_i \}$$

(the union being disjoint)

Hence
$$P\{ g(X) = y_j \} = \sum_{i : g(x_i) = y_j} P\{ X = x_i \}$$
$$= \sum_{i : g(x_i) = y_j} p(x_i).$$

Corollary: $E[aX + b] = a\,E[X] + b$

$\forall\, a, b \in \mathbb{R}$

and X is a discrete r.v.

Pf. Let $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = ax + b$.

$$\text{LHS} = E[g(X)] = \sum_{x : p(x) > 0} (ax + b)\, p(x)$$

$$= \sum_{x : p(x) > 0} ax\,p(x) + b\,p(x)$$

$$= a \sum_{x : p(x) > 0} x\,p(x) + b \sum_{x : p(x) > 0} p(x)$$

$$= a \cdot E[X] + b = \text{RHS}. \quad \blacksquare$$

Def. Let X be a discrete r.v.

For each non-negative integer n,

we call $E[X^n]$ the n-th moment of X.

§ 4.5 Variance.

Def. Let X be a discrete r.v.

Set

$$Var(X) = E[(X-\mu)^2], \quad \text{where } \mu = E[X].$$

we call $Var(X)$ the variance of X.

It describes how far X is spread out from its mean.