Math 3280A

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Review.

- Moment generating function of a r.v. X.

$$M_X(t) = E[e^{tX}], \quad t \in \mathbb{R}.$$

- If X and Y are independent,

$$M_{X+Y}(t) = M_X(t)\, M_Y(t)$$

- If $M_X(t) = M_Y(t) < \infty$ for all t in $(-a, a)$ for some $a > 0$, then X and Y have the same distribution.

Chap. 8 Limiting Thms.

§ 8.1

Q: Let $X_1, X_2, \cdots, X_n, \cdots$ be a sequence of independent, identically distributed (IID) r.v.'s.

What can one say about the limiting behavior of

$$\frac{X_1 + \cdots + X_n}{n} \quad \text{as } n \to \infty \;?$$

§ 8.2. Two basic inequalities.

Prop. 1 (Markov inequality)

Let X be a non-negative r.v. Then for any $a > 0$,

$$P\{X \geq a\} \leq \frac{E[X]}{a}.$$

Pf. Define a new r.v. I such that

$$I = \begin{cases} 1 & \text{if } X \geq a \\ 0 & \text{if } X < a. \end{cases}$$

Since $X \geq 0$, we have

$$I \leq \frac{X}{a}$$

Hence $E[I] \leq E\left[\frac{X}{a}\right] = E[X]/a.$

But
$$\begin{aligned} E[I] &= 1 \cdot P\{I=1\} + 0\, P\{I=0\} \\ &= P\{I=1\} \\ &= P\{X \geq a\}. \end{aligned}$$

Hence $$P\{X \geq a\} \leq E[X]/a. \quad \blacksquare$$

Prop. 2 (Chebyshev's inequality)

Let X be a r.v. with finite mean μ and variance σ^2. Then $\forall\ \varepsilon > 0$,

$$P\{|X-\mu| \geq \varepsilon\} \leq \frac{\sigma^2}{\varepsilon^2}.$$

Pf. Let $Y = |X-\mu|^2$. Applying Markov inequality to Y, we have

$$P\{|X-\mu| \geq \varepsilon\} = P\{Y \geq \varepsilon^2\}$$

$$\leq \frac{E[Y]}{\varepsilon^2} = \frac{E[|X-\mu|^2]}{\varepsilon^2} = \frac{\sigma^2}{\varepsilon^2}.$$

$\blacksquare$

Exer. 3. Let X be a non-negative r.v. with mean 60 and Variance 25.

(i) Estimate $P\{X > 75\}$.

(ii) Estimate $P\{40 \le X \le 60\}$.

Solution.

(i) By the Markov inequality,

$$P\{X > 75\} \le \frac{E[X]}{75} = \frac{50}{75} = \frac{2}{3}.$$

(ii) Notice that $\mu = 50$ and $\sigma^2 = 25$.

$$P\{40 \le X \le 60\} = P\{|X-\mu| \le 10\}$$

$$= 1 - P\{|X-\mu| > 10\}.$$

By the Chebyshev inequality,

$$P\{|X-\mu| > 10\} \le \frac{Var(X)}{10^2} = \frac{25}{10^2} = 0.25.$$

Hence $P\{40 \le X \le 60\} = 1 - P\{|X-\mu| > 10\}$

$$\ge 1 - 0.25 = 0.75.$$

Prop 4. Let X be a r.v. with a finite mean μ. Suppose $Var(X) = 0$. Then

$$P\{X = \mu\} = 1.$$

Proof. $$P\{X \neq \mu\} = P\left\{\bigcup_{k=1}^{\infty} |X-\mu| \geq \frac{1}{k}\right\}$$

(countable sub-additivity)

$$\leq \sum_{k=1}^{\infty} P\left\{|X-\mu| \geq \frac{1}{k}\right\}$$

(chebyshev)

$$\leq \sum_{k=1}^{\infty} \frac{Var(X)}{\left(\frac{1}{k}\right)^2} = 0$$

Hence $P\{X \neq \mu\} = 0$.

This proves $P\{X = \mu\} = 1 - P\{X \neq \mu\} = 1$. ▨

Thm 5. (The weak law of large numbers)

Let $X_1, X_2, \cdots, X_n, \cdots$ be an independent and identically distributed rv's, having a finite mean. Then for any $\varepsilon > 0$,

$$P\left\{ \left| \frac{X_1 + \cdots + X_n}{n} - \mu \right| \geq \varepsilon \right\} \to 0 \text{ as } n \to \infty.$$

Pf. We prove the thm under an additional assumption that $Var(X_i) =: \sigma^2 < \infty$.

$$E\left[\frac{X_1 + \cdots + X_n}{n}\right] = \frac{1}{n}\sum_{k=1}^{n} E[X_k]$$
$$= \mu.$$

$$Var\left(\frac{X_1 + \cdots + X_n}{n}\right) = \frac{1}{n^2} Var(X_1 + \cdots + X_n)$$
$$= \frac{1}{n^2} \cdot \sum_{k=1}^{n} Var(X_k) \quad \text{(since } X_1, \cdots, X_n \text{ are independent)}$$
$$= \frac{\sigma^2 \cdot n}{n^2} = \frac{\sigma^2}{n}.$$

Applying the Chebyshev inequality to $\frac{X_1+\cdots+X_n}{n}$,

we obtain

$$P\left\{ \left| \frac{X_1+\cdots+X_n}{n} - \mu \right| \geq \varepsilon \right\} \leq \frac{\operatorname{Var}\left(\frac{X_1+\cdots+X_n}{n}\right)}{\varepsilon^2}$$

$$= \frac{\sigma^2}{n\varepsilon^2}$$

$$\to 0 \quad \text{as } n \to \infty.$$

$\blacksquare$