Math 3280A

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Review.

- Conditional distribution.

  X, Y are jontly cts.

① $f_{X|Y}(x|y) := \dfrac{f(x,y)}{f_Y(y)}$ if $f_Y(y) > 0$

② $P\{X \in A \mid Y = y\} := \int_A f_{X|Y}(x|y)\,dx$

- Distribution of functions of r.v.'s.

Setup: X_1, X_2 are joint cts with density $f_{X_1,X_2}(x_1,x_2)$. Let $g_1, g_2 : \mathbb{R}^2 \to \mathbb{R}$.

Let $Y_1 = g_1(X_1, X_2)$, $Y_2 = g_2(X_1, X_2)$.

Find out the joint distribution of Y_1, Y_2

Thm. Assumptions: ① x_1, x_2 can be solved in terms of y_1, y_2.

② g_1, g_2 have cts partial derivatives

and
$$J(x_1, x_2) := \begin{vmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} \end{vmatrix}$$
$$= \frac{\partial g_1}{\partial x_1}\frac{\partial g_2}{\partial x_2} - \frac{\partial g_1}{\partial x_2}\frac{\partial g_1}{\partial x_1} \neq 0$$

Then Y_1, Y_2 have a joint density
$$f_{Y_1, Y_2}(y_1, y_2) = f_{X_1, X_2}(x_1, x_2) \cdot |J(x_1, x_2)|^{-1}.$$

Exer 1. X, Y have joint density
$$f(x,y) = \begin{cases} \frac{1}{x^2 y^2}, & \text{if } x>1, \ y>1 \\ 0, & \text{otherwise.} \end{cases}$$

Let $U = XY$, $V = X/Y$. Find out the joint density of U and V.

Solution: Let $g_1(x,y) = xy$ and $g_2(x,y) = x/y$.

Then

$$J(x,y) = \begin{vmatrix} \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y} \\ \frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y} \end{vmatrix} = \begin{vmatrix} y & x \\ \frac{1}{y} & -\frac{x}{y^2} \end{vmatrix}$$

$$= -\frac{2x}{y}$$

By the thm,

$$f_{U,V}(u,v) = f(x,y) \cdot |J(x,y)|^{-1}$$

$$= \begin{cases} \frac{1}{x^2y^2} \cdot \frac{y}{2x} = \frac{1}{2x^3y} & \text{if } x>1,\ y>1 \\ 0 & \text{otherwise.} \end{cases}$$

Notice that $x = \sqrt{uv}$
$y = \sqrt{u/v}$
$u, v > 0.$

$\left(\begin{array}{l} \sqrt{uv} > 1 \Rightarrow v > \frac{1}{u} \\ \sqrt{u/v} > 1 \Rightarrow v < u \end{array} \right.$

Also notice that $x, y > 1 \iff u > 1,\ \frac{1}{u} < v < u.$

Hence $f_{U,V}(u,v) = \begin{cases} \frac{1}{2u^2v}, & \text{if } u>1,\ \frac{1}{u} < v < u \\ 0, & \text{otherwise.} \end{cases}$ $\square$

Chap. 7. Properties of expectations.

§ 7.1 Introduction.

Recall that in the discrete case

$$E[X] = \sum_x x\, p(x).$$

In the cts case,

$$E[X] = \int_{-\infty}^{\infty} x f(x)\, dx$$

The expectation of X is a weighted average of the possible values of X.

§ 7.2 Expectation of functions of r.v.'s and sums of r.v.'s

Prop 2. Let $g : \mathbb{R}^2 \to \mathbb{R}$.

(1) If both X and Y are discrete with a joint prob. mass function $p(x,y)$, then

$$E[g(X,Y)] = \sum_x \sum_y g(x,y)\, p(x,y)$$

(2) If X, Y are jointly cts with a density $f(x,y)$, then

$$E[g(X,Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f(x,y)\, dx\, dy$$

Pf. Here we only prove (2) under an additional assumption that $g \geq 0$.

Recall that for a non-negative r.v. Z,

$$E[Z] = \int_0^\infty P\{Z > t\}\, dt.$$

Applying the above formula to $g(X,Y)$, we obtain

$$E[g(X,Y)] = \int_0^\infty P\{g(X,Y) > t\}\, dt$$

$$= \int_0^\infty \left(\iint_{(x,y):\, g(x,y) > t} f(x,y)\, dx\, dy \right) dt$$

$$\overset{\text{Fubini}}{=\!=} \int_{-\infty}^\infty \int_{-\infty}^\infty \left(\int_0^{g(x,y)} f(x,y)\, dt \right) dx\, dy$$

$$= \int_{-\infty}^\infty \int_{-\infty}^\infty f(x,y)\, g(x,y)\, dx\, dy.$$

$\square$

Corollary 3. $E[X+Y] = E[X] + E[Y]$.

Pf. Assume that X, Y are jointly cts with a density $f(x,y)$.

Then by Prop. 2, $E[X+Y] = \int\int_{-\infty}^{\infty} (x+y) f(x,y)\,dx\,dy$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f(x,y)\,dx\,dy + \int\int_{-\infty}^{\infty} y f(x,y)\,dx\,dy$$

$$= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} x f(x,y)\,dy\right)dx + \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} y f(x,y)\,dx\right)dy$$

$$= \int_{-\infty}^{\infty} x f_X(x)\,dx + \int_{-\infty}^{\infty} y f_Y(y)\,dy$$

$$= E[X] + E[Y]. \quad \boxtimes$$

By induction, we have the following

$$E[X_1 + \cdots + X_n] = \sum_{i=1}^{n} E[X_i].$$

§ 7.4 Covariance.

Recall the variance of a r.v. X is given by

$$\mathrm{Var}(X) = E[(X-\mu)^2], \text{ where } \mu = E[X].$$

It describes how far is X from its mean.

Def. (Covariance)

Let X, Y be two r.v.'s. The covariance of X and Y, denoted by $\mathrm{Cov}(X, Y)$, is defined by

$$\mathrm{Cov}(X, Y) = E[(X - E[X]) \cdot (Y - E[Y])].$$

In particular, $\mathrm{Cov}(X, X) = \mathrm{Var}(X)$.

Lem 4. Let X, Y be independent, and $g, h : \mathbb{R} \to \mathbb{R}$.

Then $$E[g(X) \cdot h(Y)] = E[g(X)] \cdot E[h(Y)].$$

pf. We only prove it in the cts case.

$$E[g(X)\, h(Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)\, h(y)\, f(x, y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)\, h(y)\, f_X(x)\, f_Y(y)\, dx\, dy$$

$$= \left(\int_{-\infty}^{\infty} g(x) f_X(x)\, dx\right)\left(\int_{-\infty}^{\infty} h(y) f_Y(y)\, dy\right)$$

$$= E[g(X)] \cdot E[h(Y)].$$

Corollary 5. If X, Y are independent, then $\mathrm{Cov}(X, Y) = 0$.

pf. When X, Y are independent, by Lem 4,

$$\mathrm{Cov}(X, Y) = E[(X - E[X])(Y - E[Y])]$$

$$= E[(X - E[X])] \cdot E[(Y - E[Y])]$$

$$= 0$$

$\boxed{\checkmark}$

Remark: $\mathrm{Cov}(X, Y) = 0$ does not imply that X, Y are independent.

Example 6. Let X, Y be two r.v.'s such that

① $P\{X=0\} = P\{X=-1\} = P\{X=1\} = \frac{1}{3}$

② $Y = \begin{cases} 0 & \text{if } X \neq 0. \\ 1 & \text{if } X = 0. \end{cases}$

- A short cut formula

$$\mathrm{Cov}(X, Y) = E[XY] - E[X]E[Y].$$

- $E[X] = \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot (-1) + \frac{1}{3}(1) = 0$

$P\{XY=0\} = 1 \Rightarrow E[XY] = 0 \cdot 1 = 0$

Hence $Cov(X, Y) = 0.$

But X, Y are not independent.

$$P\{X=0, Y=0\} = 0$$

But $P\{X=0\} = \frac{1}{3}$

$$P\{Y=0\} = P\{X \neq 0\}$$

$$= P\{X=1\} + P\{X=-1\}$$

$$= 2/3$$

Hence $P\{X=0, Y=0\} \neq P\{X=0\} \cdot P\{Y=0\}$

Therefore, X and Y are not independent.