Math 3280A 2022-11-3

Review

- Sums of independent r.v.'s.

(1) Both X, Y are discrete.

$$p_{X+Y}(a) = \sum_y p_X(a-y)\, p_Y(y)$$
$$= \sum_x p_Y(a-x)\, p_X(x)$$

(2) Both X and Y are cts.

$$f_{X+Y}(a) = \int_{-\infty}^{\infty} f_X(a-y)\, f_Y(y)\, dy$$
$$= \int_{-\infty}^{\infty} f_Y(a-x)\, f_X(x)\, dx$$

- Conditional distributions.

1. Suppose both X and Y are discrete.

The conditional prob. mass function of X given $Y=y$, is given by

$$p_{X|Y}(x|y) = P\{X=x \mid Y=y\} = \frac{p(x,y)}{p_Y(y)},$$

provided $p_Y(y) > 0$.

2. Suppose that X and Y are jointly cts with density $f(x,y)$.

Def. The $\boxed{\text{conditional density function of } X}$ given $Y=y$, is given by

$$f_{X|Y}(x|y) := \frac{f(x,y)}{f_Y(y)},$$

provided that $f_Y(y) > 0$.

Def. For $A \subset \mathbb{R}$, the conditional prob. of X taking values in A given $Y=y$ is given by

$$P\{X \in A \mid Y=y\} = \int_A f_{X|Y}(x|y)\, dx$$

In particular,

$$F_{X|Y}(a|y) := P\{X \leq a \mid Y=y\}$$
$$= \int_{-\infty}^{a} f_{X|Y}(x|y)\, dx.$$

Remark: If X and Y are independent,

then $f_{X|Y}(x|y) = f_X(x)$.

(since in such case $f(x,y) = f_X(x) f_Y(y)$

Remark: One may view

$$P\{X \in A \mid Y = y\}$$

$$= \lim_{\varepsilon \to 0} P\{X \in A \mid y-\varepsilon < Y < y+\varepsilon\}$$

$$= \lim_{\varepsilon \to 0} \frac{P\{X \in A,\ y-\varepsilon < Y < y+\varepsilon\}}{P\{y-\varepsilon < Y < y+\varepsilon\}}$$

Example 1. Suppose the joint density of X and Y is given by

$$f(x,y) = \begin{cases} e^{-x/y} e^{-y}/y & \text{if } x>0,\ y>0 \\ 0 & \text{otherwise} \end{cases}$$

Find $P\{X>1 \mid Y=y\}$.

Solution:

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

$$= \int_0^{\infty} e^{-x/y} e^{-y}/y \, dx \quad (\text{if } y>0)$$

$$= -e^{-x/y} e^{-y} \Big|_{x=0}^{\infty}$$

$$= e^{-y} \qquad \text{if } y>0.$$

Hence for $y>0$,

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = e^{-x/y}/y$$

if $x>0$.

Therefore

$$P\{X>1 \mid Y=y\} = \int_1^\infty e^{-x/y}/y \, dx$$

$$= -e^{-x/y}\Big|_{x=1}^{\infty}$$

$$= e^{-1/y} \quad \text{if } y>0.$$

$\blacksquare$

§ 6.7. Joint distributions of functions of r.v.'s

Setup: X_1, X_2 are joint cts with density $f_{X_1,X_2}(x_1,x_2)$. Let $g_1, g_2 : \mathbb{R}^2 \to \mathbb{R}$.

Let $Y_1 = g_1(X_1, X_2)$, $Y_2 = g_2(X_1, X_2)$.

Find out the joint distribution of Y_1, Y_2

Thm 2. Assumptions: ① x_1, x_2 can be solved in terms of y_1, y_2.

② g_1, g_2 have cts partial derivatives

and
$$J(x_1, x_2) := \begin{vmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} \end{vmatrix}$$
$$= \frac{\partial g_1}{\partial x_1}\frac{\partial g_2}{\partial x_2} - \frac{\partial g_1}{\partial x_2}\frac{\partial g_1}{\partial x_1} \neq 0$$

Then Y_1, Y_2 have a joint density
$$f_{Y_1, Y_2}(y_1, y_2) = f_{X_1, X_2}(x_1, x_2) \cdot |J(x_1, x_2)|^{-1}$$

Example 3. Let X_1, X_2 be jointly cts with density $f(x_1, x_2)$.

Let $Y_1 = X_1 + X_2$, $Y_2 = X_1 - X_2$.

Find the joint density of Y_1, Y_2.

Solution: Let $g_1(x_1, x_2) = x_1 + x_2$

$g_2(x_1, x_2) = x_1 - x_2$.

- If $y_1 = x_1 + x_2$, $y_2 = x_1 - x_2$, then $\begin{cases} x_1 = \dfrac{y_1 + y_2}{2} \\ x_2 = \dfrac{y_1 - y_2}{2} \end{cases}$

- $$J(x_1, x_2) = \begin{vmatrix} \dfrac{\partial g_1}{\partial x_1} & \dfrac{\partial g_1}{\partial x_2} \\ \dfrac{\partial g_2}{\partial x_1} & \dfrac{\partial g_2}{\partial x_2} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}$$

$$= -2.$$

- $$f_{Y_1, Y_2}(y_1, y_2) = f(x_1, x_2) \cdot |J(x_1, x_2)|^{-1}$$

$$= f(x_1, x_2) / 2$$

$$= f\left(\frac{y_1 + y_2}{2}, \frac{y_1 - y_2}{2}\right) / 2.$$

$\square$