Math 3280 22-09-19

Review

- Sample spaces having equally likely outcomes.

$$P(E) = \frac{\# E}{\# S}.$$

Exer 1. A deck of 52 cards in dealt out. What is the probability that the first ace occurs in the 14th card.

Solution: Let E denote the event that the first ace occurs in the 14 th card. Let S denote the sample space.

Then $\# S = 52!$

$$\# E = 48 \times 47 \times \cdots \times 36 \times 4 \times (38!)$$

Hence

$$P(E) = \frac{\# E}{\# S} = \frac{48 \times 47 \times \cdots \times 36 \times 4}{52 \times 51 \times \cdots \times 39}.$$

Chap 3. Conditional probability and independence

§ 3.1 Conditional probability.

Example 1: Let us roll two dices. Suppose the first die is a 3. Given this information, what is the prob. that the sum of 2 dices equals 8?

Sol: F — the event that the first die is 3

E — the event that the sum of 2 dices equals 8.

$$F = \{(3,1), (3,2), (3,3), (3,4), (3,5), (3,6)\}$$

$$E = \{(i,j) \in \{1,2,3,4,5,6\}^2 : i+j = 8\}.$$

$$E \cap F = \{(3,5)\}.$$

The prob. of each outcome in F is $\frac{1}{6}$, so is the outcome $\{(3,5)\}$.

Hence the (conditional) prob of E given F is $\frac{1}{6}$.

Def. (Conditional prob.).

Let E, F be two events for a random experiment. Suppose $P(F) > 0$. Then the conditional prob. of E given F is

$$P(E|F) = \frac{P(EF)}{P(F)}.$$

Remark: If $P(F) = 0$,

then $P(E|F)$ is not well-defined.

But in practice, you may assign any value from $[0,1]$ to $P(E|F)$.

Prop. (Multiplicative rule)

- $P(E_1 E_2) = P(E_1)\, P(E_2 | E_1)$
- $P(E_1 E_2 \cdots E_n)$

$$= P(E_1) \cdot P(E_2|E_1) \cdot P(E_3|E_1 E_2) \cdots \cdot P(E_n | E_1 E_2 \cdots E_{n-1})$$

Pf. Since $P(E_2|E_1) = \dfrac{P(E_1 E_2)}{P(E_1)}$, so

$$P(E_1 E_2) = P(E_1) \cdot P(E_2|E_1).$$

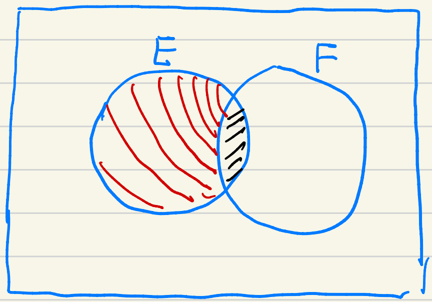
To see the second identity,

$$\text{RHS} = P(E_1) \cdot \frac{P(E_1 E_2)}{P(E_1)} \cdot \frac{P(E_1 E_2 E_3)}{P(E_1 E_2)} \cdots \frac{P(E_1 \cdots E_n)}{P(E_1 \cdots E_{n-1})}$$

$$= P(E_1 \cdots E_n). \quad \blacksquare$$

§ 3.2 Bayes' formula.

Let E, F be two events.



$$E = (E \cap F) \cup (E \cap F^c)$$

(black). (red)

Hence
$$P(E) = P(E \cap F) + P(E \cap F^c)$$

But
$$P(E \cap F) = P(F) \cdot P(E|F),$$
$$P(E \cap F^c) = P(F^c) \cdot P(E|F_c).$$

We obtain

$$P(E) = P(F)\cdot P(E|F) + P(F^c)\cdot P(E|F^c).$$

(Total probability formula).

Hence to determine the prob. of E, we may first conduct the "conditioning" upon whether or not the event F has occured.

Next we give a generenization of this formula.

Let $F_1, F_2, \cdots, F_n$ be a sequence of events such that they are mutually exclusive, and $\bigcup_{k=1}^{n} F_k = S$ (we say $F_1, \cdots, F_n$ are exhaustive)

Then we have

$$P(E) = \sum_{k=1}^{n} P(F_k) \cdot P(E|F_k).$$

Pf: Notice that $E = \bigcup_{k=1}^{n} (E \cap F_k)$

(with disjoint union)

Hence

$$P(E) = \sum_{k=1}^{n} P(E \cap F_k)$$

$$= \sum_{k=1}^{n} P(F_k) P(E|F_k). \quad \square$$

Prop. (Bayes' formula).

Assume $F_1, \cdots, F_n$ are mutually exclusive and exhaustive.

Then for any $1 \le i \le n$,

$$P(F_i|E) = \frac{P(F_i) \cdot P(E|F_i)}{\sum_{k=1}^{n} P(F_k) P(E|F_k)}$$

Pf:

$$\sum_{k=1}^{n} P(F_k) P(E|F_k) = P(E)$$

$$P(F_i) \cdot P(E|F_i) = P(EF_i)$$

$\blacksquare$