

THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH3280 Introductory Probability 2022-2023 Term 1
Suggested Solutions of Homework Assignment 5

Q1

(a) Let $p(x_1, x_2)$ be the joint probability mass X_1 and X_2 .

$$\begin{aligned} p(0, 0) &= \frac{8}{13} \cdot \frac{7}{12} = \frac{14}{39} \\ p(0, 1) &= \frac{8}{13} \cdot \frac{5}{12} = \frac{10}{39} \\ p(1, 0) &= \frac{5}{13} \cdot \frac{8}{12} = \frac{10}{39} \\ p(1, 1) &= \frac{5}{13} \cdot \frac{4}{12} = \frac{5}{39} \end{aligned}$$

(b) Let $q(x_1, x_2, x_3)$ be the joint probability mass of X_1, X_2 and X_3 .

$$\begin{aligned} q(0, 0, 0) &= \frac{8}{13} \cdot \frac{7}{12} \cdot \frac{6}{11} = \frac{28}{143} & q(0, 1, 1) &= \frac{8}{13} \cdot \frac{5}{12} \cdot \frac{4}{11} = \frac{40}{429} \\ q(0, 0, 1) &= \frac{8}{13} \cdot \frac{7}{12} \cdot \frac{5}{11} = \frac{70}{429} & q(1, 1, 0) &= \frac{5}{13} \cdot \frac{4}{12} \cdot \frac{8}{11} = \frac{40}{429} \\ q(0, 1, 0) &= \frac{8}{13} \cdot \frac{5}{12} \cdot \frac{7}{11} = \frac{70}{429} & q(1, 0, 1) &= \frac{5}{13} \cdot \frac{8}{12} \cdot \frac{4}{11} = \frac{40}{429} \\ q(1, 0, 0) &= \frac{5}{13} \cdot \frac{8}{12} \cdot \frac{7}{11} = \frac{70}{429} & q(1, 1, 1) &= \frac{5}{13} \cdot \frac{4}{12} \cdot \frac{3}{11} = \frac{5}{143} \end{aligned}$$

Q2

(a) Since f is non-negative and

$$\iint_{\mathbb{R}^2} f(x, y) dx dy = \int_0^1 \int_0^2 \frac{6}{7} (x^2 + xy/2) dy dx = 1$$

it follows that f is a joint density function.

(b) The density of X is given by

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y) dy = \int_0^2 \frac{6}{7} (x^2 + xy/2) dy = \frac{6}{7} (2x^2 + x), \quad x \in (0, 1)$$

and $f_X(x) = 0$ elsewhere.

(c)

$$P(X > Y) = \iint_{\{(x,y):x>y\}} f(x,y)dxdy = \int_0^1 \int_0^x \frac{6}{7} (x^2 + xy/2) dydx = \frac{15}{56}$$

(d)

$$\begin{aligned} P(Y > 1/2 | X < 1/2) &= \frac{P(X < 1/2, Y > 1/2)}{P(X < 1/2)} \\ &= \frac{\int_0^{1/2} \int_{1/2}^2 \frac{6}{7} (x^2 + xy/2) dydx}{\int_0^{1/2} \int_0^2 \frac{6}{7} (x^2 + xy/2) dydx} \\ &= \frac{\frac{6}{7} \cdot \frac{23}{128}}{\frac{6}{7} \cdot \frac{5}{24}} \\ &= \frac{69}{80} \end{aligned}$$

(e)

$$E(X) = \int_{-\infty}^{\infty} x \cdot f_X(x)dx = \int_0^1 x \cdot \frac{6}{7} (2x^2 + x) dx = \frac{5}{7}$$

(f) The density of Y is given by

$$f_Y(y) = \int_0^1 \frac{6}{7} (x^2 + xy/2) dx = \frac{6}{7} \left(\frac{1}{3} + \frac{y}{4} \right), \quad y \in (0, 2)$$

and $f_Y(y) = 0$ elsewhere. Thus

$$E(Y) = \int_{-\infty}^{\infty} y \cdot f_Y(y)dy = \int_0^2 y \cdot \frac{6}{7} \left(\frac{1}{3} + \frac{y}{4} \right) dy = \frac{8}{7}$$

Q3

(a)

$$P(X < Y) = \iint_{\{(z,y):x<y\}} f(x,y)dxdy = \int_0^{\infty} \int_x^{\infty} e^{-(x+y)} dydx = \frac{1}{2}$$

(b)

$$P(X < a) = \begin{cases} \int_0^a \int_0^{\infty} e^{-(x+y)} dydx = 1 - e^{-a} & a > 0 \\ 0 & a \leq 0 \end{cases}$$

Q4

(a) The density of X and Y are given by

$$f_X(x) = \begin{cases} \int_0^1 (x+y)dy = x + \frac{1}{2} & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_Y(y) = \begin{cases} \int_0^1 (x+y)dx = y + \frac{1}{2} & 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Since $f \neq f_X \cdot f_Y$, we conclude that X and Y are not independent.

(b) See $f_X(x)$ in (a).

(c)

$$P(X + Y < 1) = \iint_{\{(z,y):x+y<1\}} f(x,y)dx dy = \int_0^1 \int_0^{1-x} (x+y)dy dx = \frac{1}{3}$$

Q5

(a) The joint density of A, B and C is given by $f(a, b, c) = f_A(a) \cdot f_B(b) \cdot f_C(c)$. Thus the joint cumulative distribution of A, B and C is

$$F(a, b, c) = \int_{-\infty}^a \int_{-\infty}^b \int_{-\infty}^e f_A(a) \cdot f_B(b) \cdot f_C(c) dc db da = F_A(a) \cdot F_B(b) \cdot F_C(c)$$

where

$$F_A(t) = F_B(t) = F_C(t) = \begin{cases} 1, & t \geq 1 \\ t, & 0 < t < 1 \\ 0, & t \leq 0 \end{cases}$$

(b) Note that all roots of $Ax^2 + Bx + C$ are real if and only if $B^2 \geq 4AC$.

$$\begin{aligned} P(B^2 \geq 4AC) &= \iiint_{\{(a,b,c) \in [0,1]^3: b^2 \geq 4ac\}} f(a,b,c) da db dc \\ &= \int_0^{1/4} \int_0^1 \int_{\sqrt{4ac}}^1 db dc da + \int_{1/4}^1 \int_0^1 \int_0^{\frac{1}{4a}b^2} dc db da \\ &= \frac{5}{36} + \frac{1}{6} \ln 2 \end{aligned}$$

where the second equality is derived by the following argument:

- If $0 \leq a \leq 1/4$, then $4ac \leq 1$ always hold for $0 \leq c \leq 1$, thus $\sqrt{4ac} \leq b \leq 1$

- If $1/4 \leq a \leq 1$, then $b^2/4a \leq 1$ always hold for $0 \leq b \leq 1$, thus $0 \leq c \leq b^2/4a$

Alternatively,

$$\begin{aligned} P(B^2 \geq 4AC) &= 1 - \iiint_{\{(a,b,c) \in [0,1]^3: b^2 \leq 4ac\}} f(a,b,c) da db dc \\ &= 1 - \int_0^1 \int_{b^2/4}^1 \int_{b^2/4a}^1 dc da db \\ &= \frac{5}{36} + \frac{1}{6} \ln 2 \end{aligned}$$

Q6

(a) Let $g_1(x, y) = x + y$ and $g_2(x, y) = x/y$. Then

$$|J(x, y)| = \begin{vmatrix} \frac{\partial g_1(x,y)}{\partial x} & \frac{\partial g_1(x,y)}{\partial y} \\ \frac{\partial g_2(x,y)}{\partial x} & \frac{\partial g_2(x,y)}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1/y & -x/y^2 \end{vmatrix} = \left| \frac{x+y}{y^2} \right|$$

Note that

$$\begin{cases} U = X + Y \\ V = \frac{X}{Y} \end{cases} \Leftrightarrow \begin{cases} X = \frac{UV}{V+1} \\ Y = \frac{U}{V+1} \end{cases}$$

Hence the joint density of U and V is

$$\begin{aligned} f_{U,V}(u, v) &= f(x, y) \cdot |J(x, y)|^{-1} \\ &= f\left(\frac{uv}{v+1}, \frac{u}{v+1}\right) \cdot \left| J\left(\frac{uv}{v+1}, \frac{u}{v+1}\right) \right|^{-1} \\ &= \begin{cases} \frac{u}{(v+1)^2} & 0 < \frac{u}{v+1} < 1, 0 < \frac{uv}{v+1} < 1 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

(b) Let $g_1(x, y) = x$ and $g_2(x, y) = x/y$. Then

$$|J(x, y)| = \begin{vmatrix} 1 & 0 \\ 1/y & -x/y^2 \end{vmatrix} = \left| \frac{x}{y^2} \right|$$

Note that

$$\begin{cases} U = X \\ V = \frac{X}{Y} \end{cases} \Leftrightarrow \begin{cases} X = U \\ Y = \frac{U}{V} \end{cases}$$

Hence the joint density of U and V is

$$\begin{aligned} f_{U,V}(u, v) &= f(x, y) \cdot |J(x, y)|^{-1} \\ &= f\left(u, \frac{u}{v}\right) \cdot \left|J\left(u, \frac{u}{v}\right)\right|^{-1} \\ &= \begin{cases} \frac{u}{v^2} & 0 < u < 1, 0 < \frac{u}{v} < 1 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

(c) Let $g_1(x, y) = x + y$ and $g_2(x, y) = \frac{x}{x+y}$. Then

$$|J(x, y)| = \left| \begin{array}{cc} 1 & 1 \\ \frac{y}{(x+y)^2} & \frac{-x}{(x+y)^2} \end{array} \right| = \left| \frac{1}{x+y} \right|$$

Note that

$$\begin{cases} U = X + Y \\ V = \frac{X}{X+Y} \end{cases} \Leftrightarrow \begin{cases} X = UV \\ Y = U - UV \end{cases}$$

Hence the joint density of U and V is

$$\begin{aligned} f_{U,V}(u, v) &= f(x, y) \cdot |J(x, y)|^{-1} \\ &= f(uv, u - uv) \cdot |J(u, u - uv)|^{-1} \\ &= \begin{cases} u & 0 < uv < 1, 0 < u - uv < 1 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Q7

(a) We have $P(X = j, Y = k) = P(Y = k | X = j)P(X = j) = \frac{1}{5j}$. Since given $X = j$ we know Y is uniform in $\{1, 2, \dots, j\}$. Here we have $k \leq j$. Thus

$$P(Y = 1) = \sum_{j=1}^5 \frac{1}{5j} = \frac{137}{300} = c$$

(b) We use Bayes formula to get

$$P(X = j | Y = 1) = P(Y = 1 | X = j) \frac{P(X = j)}{P(Y = 1)} = \frac{1}{5j} \frac{1}{c}$$

where c as in (1).

(c) Not independent: $P(X = 1, Y = 1) = \frac{1}{5}$. But $P(X = 1)P(Y = 1) = \frac{c}{5} \neq \frac{1}{5}$.

Q8

(a) For $x > 0$, we have that

$$f_X(x) = \int_0^{\infty} f(x, y) dy = \int_0^{\infty} x e^{-x(y+1)} dy = -e^{-x(y+1)} \Big|_{y=0}^{y=\infty} = e^{-x}$$

and for $y > 0$ we have that

$$f_Y(y) = \int_0^{\infty} f(x, y) dx = \int_0^{\infty} x e^{-x(y+1)} dx$$

In order to solve this integral, use integration by parts. Define $u = x$ and $dv = e^{-x(y+1)} dx$. Thus

$$f_Y(y) = -\frac{x}{y+1} \cdot e^{-x(y+1)} \Big|_0^{\infty} + \frac{1}{y+1} \int_0^{\infty} e^{-x(y+1)} dx = \frac{1}{(y+1)^2}$$

Now, for $x > 0$, the conditional density of X given $Y = y$ is

$$f_{X|Y}(x | y) = \frac{f(x, y)}{f_Y(y)} = \frac{x e^{-x(y+1)}}{\frac{1}{(y+1)^2}} = x(y+1)^2 \cdot e^{-x(y+1)}$$

If $x \leq 0$, then $f_{X|Y}(x | y) = 0$. And similarly, for $y > 0$,

$$f_{Y|X}(y | x) = \frac{f(x, y)}{f_X(x)} = \frac{x e^{-x(y+1)}}{e^{-x}} = x e^{-xy}$$

For $y \leq 0$, $f_{Y|X}(y | x) = 0$.

(b) If $z \leq 0$, then $F_Z(z) = 0$, which implies $f_Z(z) = 0$. If $z > 0$, we have

$$P(Z \leq z) = P(XY \leq z) = \int_0^{\infty} \int_0^{\frac{z}{x}} f(x, y) dy dx$$

Take the derivative with respect to z , we then get

$$f_Z(z) = \int_0^{\infty} \frac{1}{x} f(x, \frac{z}{x}) dx = \int_0^{\infty} e^{-(x+z)} dx = e^{-z}$$

Q9

Let $f(x), g(y)$ be the densities of X, Y . (a) Since the cumulative distribution of Z is

$$F_Z(z) = P(X/Y \leq z) = P(X \leq Yz) = \int_0^{\infty} \int_0^{yz} f(x)g(y) dx dy$$

then the density of Z is

$$h(z) = \frac{dF_Z(z)}{dz} = \int_0^\infty \frac{d}{dz} \int_0^{yz} f(x)g(y)dx dy = \int_0^\infty y f(yz)g(y)dy$$

When $z \leq 0$, $h(z) = 0$. (b) Since the cumulative distribution of Z is

$$F_Z(z) = P(XY \leq z) = P(X \leq z/Y) = \int_0^\infty \int_0^{z/y} f(x)g(y)dx dy$$

then the density of Z is

$$h(z) = \frac{dF_Z(z)}{dz} = \int_0^\infty \frac{d}{dz} \int_0^{z/y} f(x)g(y)dx dy = \int_0^\infty \frac{1}{y} f\left(\frac{z}{y}\right) g(y)dy$$

When $z \leq 0$, $h(z) = 0$ If $f(x) = \lambda \exp(-\lambda x)$, $x > 0$ and $g(y) = \eta \exp(-\eta y)$, $y > 0$ for some $\lambda, \eta > 0$, then (a)

$$h(z) = \begin{cases} \lambda\eta \int_0^\infty ye^{-(\lambda z + \eta)y} dy = \frac{\lambda\eta}{(z\lambda + \eta)^2} & z > 0 \\ 0 & \text{otherwise} \end{cases}$$

(b)

$$h(z) = \begin{cases} \lambda\eta \int_0^\infty \frac{1}{y} e^{-(\lambda z/y + \eta y)} dy & z > 0 \\ 0 & \text{otherwise} \end{cases}$$

Q10

$$\begin{aligned} P(X = n, Y = m) &= P(X_1 + X_2 = n, X_2 + X_3 = m) \\ &= \sum_{k=0}^{\min\{n,m\}} P(X_1 = n - k, X_2 = k, X_3 = m - k) \\ &= \sum_{k=0}^{\min\{n,m\}} P(X_1 = n - k) P(X_2 = k) P(X_3 = m - k) \\ &= \sum_{k=0}^{\min\{n,m\}} \frac{e^{-\lambda_1} \lambda_1^{n-k}}{(n-k)!} \cdot \frac{e^{-\lambda_2} \lambda_2^k}{k!} \cdot \frac{e^{-\lambda_3} \lambda_3^{m-k}}{(m-k)!} \\ &= e^{-(\lambda_1 + \lambda_2 + \lambda_3)} \sum_{k=0}^{\min\{n,m\}} \frac{\lambda_1^{n-k} \lambda_2^k \lambda_3^{m-k}}{(n-k)! \cdot k! \cdot (m-k)!} \end{aligned}$$