Topic#19
Spectral decomposition

Proposition. Let V be an i.p.s. and $W \subset V$ be a finite-dim subspace with an orthonormal basis $\{v_1, \cdots, v_k\}$. Then the **orthogonal projection** $T : V \to V$ defined by

$$T(y) = \sum_{i=1}^{k} \langle y, v_i \rangle v_i,$$

is a linear operator s.t.

(a) $N(T) = W^\perp$ and $R(T) = W$.

(b) $T^2 = T$.

(c) T is self-adjoint.

RK: In fact, properties (a) and (b) **uniquely define** the orthogonal projection onto W, so they are also often used as the definition of an orthogonal projection.

Pf.: First note that T is linear because $\langle \cdot, \cdot \rangle$ is linear in the first component.

(a) Note

$$\begin{aligned} N(T) &= \{y \in V : \sum_{i=1}^{k} \langle y, v_i \rangle v_i = 0\} \\ &= \{y \in V : \langle y, v_i \rangle = 0, i = 1, \cdots, k\} \\ &= W^{\perp}, \end{aligned}$$

since $\{v_1, \cdots, v_k\}$ is a basis for W.

To show: $R(T) = W$.
By definition, $R(T) \subset W$. On the other hand, let $u \in W$,

Note $W = span(\{v_1, \cdots, v_n\})$ and $\{v_1, \cdots, v_n\}$ is orthonormal.
We have:

$$u = \sum_{i=1}^{k} \langle u, v_i \rangle v_i = T(u),$$

so $W \subset R(T)$. Thus, $R(T) = W$, and $T|_W = I_W$.

(b) From (a), we see that

$$T^2 = T \circ T = T|_{R(T)} \circ T = T|_W \circ T = I_W \circ T = T.$$

(c) Take $x, y \in V = W \oplus W^\perp$, then

$$x = x_1 + x_2, \quad y = y_1 + y_2$$

with $x_1, y_1 \in W$ and $x_2, y_2 \in W^\perp$. Then,

$$T(x) = x_1,\ T(y) = y_1.$$

Hence,

$$\langle T(x), y\rangle = \langle x_1, y_1 + y_2\rangle = \langle x_1, y_1\rangle,$$
$$\langle x, T(y)\rangle = \langle x_1 + x_2, y_1\rangle = \langle x_1, y_1\rangle.$$

So it holds that $\langle T(x), y\rangle = \langle x, T(y)\rangle$. This shows $T = T^*$, i.e. T is self-adjoint. □

Spectral Theorem. Let T be a linear operator on a finite-dim i.p.s. V over F with distinct eigenvalues $\lambda_1, \cdots, \lambda_k$. Assume that T is normal (resp. self-adjoint) if $F = \mathbb{C}$ (resp. $F = \mathbb{R}$). For $i = 1, \cdots, k$, let $E_i = E_{\lambda_i}$ be the eigenspace of T corresponding to λ_i, and let T_i be the orthogonal projection onto E_i. Then,

(a) $V = E_1 \oplus E_2 \oplus \cdots \oplus E_k$.

(b) $E_i^{\perp} = \oplus_{j \neq i} E_j$ for $i = 1, \cdots, k$.

(c) $T_i T_j = \delta_{ij} T_j$ for $1 \leq i, j \leq k$.

(d) $I = T_1 + T_2 + \cdots + T_k$. (resolution of identity)

(e) $T = \lambda_1 T_1 + \lambda_2 T_2 + \cdots + \lambda_k T_k$. (spectral decomposition)

Pf.: (a) This follows from the fact that T is diagonalizable.

(b) We already know that $E_j \subset E_i^{\perp}$ for $j \neq i$, so $\oplus_{j\neq i} E_j \subset E_i^{\perp}$. The identity then follows by comparing the dimensions:

$$\dim(E_i^{\perp}) = \dim(V) - \dim(E_i) = \sum_{j\neq i} \dim(E_j).$$

(c) It is direct to see

$$T_i T_j = T_i|_{R(T_j)} \circ T_j = T_i|_{E_j} \circ T_j = \delta_{ij} I_{E_j} \circ T_j = \delta_{ij} T_j.$$

(d)&(e): Since $V = E_1 \oplus \cdots \oplus E_k$, any $x \in V$ can be expressed uniquely as

$$x = x_1 + x_2 + \cdots + x_k, \quad x_i \in E_i.$$

Then $T_i(x) = T_i(x_1) + \cdots + T_i(x_k) = T_i(x_i) = x_i$ since T_i is orthogonal projection on E_{λ_i}. Then $(T_1 + \cdots + T_k)(x) = T_1(x) + \cdots + T_k(x) = x_1 + \cdots + x_k = x = I(x)$, showing (d). Further, we see: $T(x) = T(x_1) + \cdots + T(x_k) = \lambda_1 x_1 + \cdots + \lambda_k x_k = \lambda_1 T_1(x) + \cdots + \lambda_k T_k(x) = (\lambda_1 T_1 + \cdots + \lambda_k T_k)(x)$, showing (e). □

RK: The set

$$\{\lambda_1, \cdots, \lambda_k\}$$

of distinct eigenvalues of T is called the **spectrum** of T; the decomposition

$$I = T_1 + \cdots + T_k$$

is called the **resolution of the identity operator** induced by T; and

$$T = \lambda_1 T_1 + \cdots + \lambda_k T_k$$

is call the **spectral decomposition** of T, which says that, w.r.t. an orthonormal basis β of eigenvectors of T, we have

$$[T]_\beta = \begin{pmatrix} \lambda_1 I_{m_1} & 0 & \cdots & 0 \\ 0 & \lambda_2 I_{m_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_k I_{m_k} \end{pmatrix},$$

where $m_i = \dim(E_{\lambda_i}) \geq 1$ and $m_1 + ... + m_k = \dim(V)$.

We give an application:

Proposition. If $F = \mathbb{C}$ then T is normal **iff**

$$T^* = g(T) \text{ for some polynomial } g.$$

Pf.: $\Leftarrow$: Let $T^* = g(T)$ for a polynomial g, for instance,

$$g(z) = \sum_{i=1}^{n} c_i z^i,$$

then $T^* = g(T) = \sum_{i=1}^{n} c_i T^i$. Notice

$$g(T) \circ T = \sum_{i=1}^{n} c_i T^{i+1} \text{ and } T \circ g(T) = \sum_{i=1}^{n} c_i T^{i+1}.$$

This gives

$$T^* T = g(T)T = Tg(T) = TT^*,$$

so T is normal. □

$\Rightarrow$: Since $F = \mathbb{C}$, the c.p. of T splits over $\mathbb{C}$. Let $\lambda_1, \cdots, \lambda_k$ be all distinct e-values. Applying the spectrum theorem for the normal operator T, we have $T = \lambda_1 T_1 + ... + \lambda_k T_k$, where each T_i is the eigenprojection which is self-adjoint. To proceed further, we recall:

> **Lagrange interpolation formula**: For distinct complex numbers $\lambda_1, \cdots, \lambda_k$, there exists a polynomial g with $\deg g = k-1$ such that $g(\lambda_j) = \overline{\lambda_j}$, where $\overline{\lambda_j}$ is the conjugate of λ_j. Indeed, g can be given by $g = \sum_{i=1}^k g_i$ with $g_j(x) = \overline{\lambda_j} \prod_{i=1, i\neq j}^{k} \frac{x-\lambda_i}{\lambda_j - \lambda_i}$, $j = 1, ..., k$. Note: $g_j(\lambda_l) = \overline{\lambda_j}\delta_{jl}$.

Therefore,

$$\begin{aligned} g(T) &= g(\lambda_1 T_1 + \cdots + \lambda_k T_k) \\ &= g(\lambda_1) T_1 + \cdots + g(\lambda_k) T_k \text{ (Why? Exercise!)} \\ &= \overline{\lambda_1} T_1^* + \cdots + \overline{\lambda_k} T_k^* \\ &= (\lambda_1 T_1 + \cdots + \lambda_k T_k)^* \\ &= T^*. \quad \square \end{aligned}$$