Topic#18
Unitary operator & othogonal operator

Recall: Let $A \in M_{n\times n}(\mathbb{C})$ be normal, i.e. $AA^* = A^*A$, then

$$[L_A]_\beta = \mathrm{diag}(\lambda_1, \cdots, \lambda_n)$$

where $\beta = \{v_1, \cdots, v_n\}$ is an orthonormal o.b. for $\mathbb{C}^n$ consisting of e-vectors of L_A. On the other hand, we also have

$$[L_A]_\beta = [I \circ L_A \circ I]_\beta = [I]_\gamma^\beta [L_A]_\gamma^\gamma [I]_\beta^\gamma = Q^{-1}AQ$$

where $Q = (v_1 | \cdots | v_n)$ and γ is the s.o.b. .

Claim: $QQ^* = I_n = Q^*Q$ → we say such Q is a unitary matrix

Proof: For instance, $(Q^*Q)_{ij} = \sum_{l=1}^n (Q^*)_{il} Q_{lj} = \sum_{l=1}^n \bar{Q}_{li} Q_{lj}$
$= \bar{v}_i \cdot v_j = \langle v_j, v_i \rangle = \delta ij$

$\therefore Q^*Q = I_n \therefore Q^{-1} = Q^*$

$\therefore QQ^* = QQ^{-1} = I_n$

$U(n) \overset{def}{=} \{$ all unitary matrices: $QQ^* = I_n = Q^*Q, Q \in M_{n\times n}(\mathbb{F})$ $\}$

Rmk: If $Q \in U(n)$, then $Q^{-1} = Q^*$.

We have showed that if $A \in M_{n\times n}(\mathbb{C})$ is normal then $\exists Q \in U(n)$ s.t. Q^*AQ is diagonal. In this case, we say: A is unitarily equivalent to a diagonal matrix.

Theorem. $A \in M_{n\times n}(\mathbb{C})$ is normal **iff** A is unitarily equivalent to a diagonal matrix, i.e. $\exists Q \in U(n)$ s.t. Q^*AQ is diagonal.

Pf.: $\Rightarrow$: showed before.

$\Leftarrow$: Assume that $\exists\, P \in U(n)$ s.t. $P^*AP := D$ is diagonal, then

$$A = (P^*)^{-1}DP^{-1} = PDP^*, A^* = (PDP^*)^* = PD^*P^*.$$

Check:

$$AA^* = (PDP^*)(PD^*P^*) = PDD^*P^*,$$
$$A^*A = (PD^*P^*)(PDP^*) = PD^*DP^*.$$

As $D \in M_{n\times n}(\mathbb{C})$ is diagonal, D is normal, i.e.

$$DD^* = D^*D.$$

Plug it back, one has $AA^* = A^*A$, so A is normal. □

In the same way:

Let $A \in M_{n\times n}(\mathbb{R})$ be self-adjoint, i.e. A is real symmeric, then

$$[L_A]_\beta = \mathrm{diag}(\lambda_1, \cdots, \lambda_n)$$

where $\beta = \{v_1, \cdots, v_n\}$ is an orthonormal o.b. for $\mathbb{R}^n$ consisting of e-vectors of L_A. On the other hand, one also has

$$[L_A]_\beta = [I \circ L_A \circ I]_\beta = [I]_\gamma^\beta [L_A]_\gamma^\gamma [I]_\beta^\gamma = Q^{-1}AQ$$

where $Q = (v_1|\cdots|v_n)$ and γ is the s.o.b.

Claim: $Q^tQ = I_n = QQ^t$ (Exercise)
Then, $Q^{-1} = Q^t = Q^*$, $\therefore$ Q^*AQ is diagonal.
$O(n) \overset{def}{=} \{$ all orthogonal matrices: $Q^tQ = I_n = QQ^t$ $\}$

> **Theorem.** $A \in M_{n\times n}(\mathbb{R})$ is self-adjoint (i.e. real symmetric) **iff** A is orthogonally equivalent to a diagonal matrix, i.e. $\exists P \in O(n)$ s.t. P^*AP is diagonal.

Extend it to $T \in \mathcal{L}(V)$ where V is i.p.s, $F = \mathbb{C}$ or $\mathbb{R}$, n=dim(V)$<\infty$.

Def.: Let $T \in \mathcal{L}(V)$ be normal where V is a finite-dim i.p.s. over F. If

$$TT^* = I = T^*T$$

we say that the normal operator T is

- a **unitary operator** for $F = \mathbb{C}$, and
- an **orthogonal operator** for $F = \mathbb{R}$.

Example: Let $Q = (v_1|\cdots|v_n) \in M_{n\times n}(F)$, where $\beta = \{v_1, \cdots, v_n\}$ is an orthonormal basis for F^n.

Then $Q \overset{def}{=} [I_n]_\beta^\gamma = (v_1|\cdots|v_n)$ Show that

if $F = \mathbb{C}$ then

$$Q^*Q = QQ^* = I_n.$$

if $F = \mathbb{R}$ then

$$Q^tQ = QQ^t = I_n.$$

Hint: One can show that if $F = \mathbb{C}$,

$$v_i \cdot \overline{v_j} = \delta_{ij},$$

and if $F = \mathbb{R}$,

$$v_i \cdot v_j = \delta_{ij}.$$

Theorem. Let $T \in \mathcal{L}(V)$, where V is a finite-dim i.p.s over $\mathbb{F}$. Then, the following statements are equivalent:

(a) $TT^* = T^*T = I$

(b) T preserves the inner product on V, i.e.

$$\langle T(x), T(y) \rangle = \langle x, y \rangle, \ \forall x, y \in v$$

(c) If β is an orthonormal basis for V, then $T(\beta)$ is an orthonormal basis for V.

(d) $\exists$ an orthonormal basis for V s.t. $T(\beta)$ is an orthonormal basis for V

(e) $\|T(x)\| = \|x\|, \ \forall x \in V$

Remark. One may take one of items (a)-(e) as definition of unitary or orthogonal operators in terms of $F = \mathbb{C}$ or $\mathbb{R}$, respectively.

Proof.

$(a) \Rightarrow (b)$: $\langle T(x), T(y)\rangle = \langle x, T^*T(y)\rangle = \langle x, I(y)\rangle = \langle x, y\rangle$.

$(b) \Rightarrow (c)$: Let $\beta = \{v_1, \cdots, v_n\}$ be an orthonormal basis for V.
Then, $\langle T(v_i), T(v_j)\rangle = \langle v_i, v_j\rangle = \delta_{ij}$
$\therefore T(\beta) = \{T(v_1), \cdots, T(v_n)\}$ is an orthonormal basis for V.

$(c) \Rightarrow (d)$: obvious n=dim(V)$<\infty \Rightarrow \exists$ an orthonormal basis for V.

$(d) \Rightarrow (e)$: take $x \in V$. Let $\beta = \{v_1, \cdots, v_n\}$ be an orthonormal basis for V such that $T(\beta) = \{T(v_1), \cdots, T(v_n)\}$ is also an orthonormal basis for V. Then $x = \sum_{i=1}^n a_i v_i$ for some $a_1, \cdots, a_n \in \mathbb{F}$. Then $\|x\|^2 = \sum_{i=1}^n |a_i|^2$.
$\|T(x)\|^2 = \|\sum_{i=1}^n a_i T(v_i)\|^2 = \sum_{i=1}^n |a_i|^2$. Hence, $\|T(x)\| = \|x\|$.

$(e) \Rightarrow (a)$: to show $U \overset{def.}{=} I - T^*T$ is zero operator. Indeed, let $x \in V$, then by (e)
$\langle x, (I - T^*T)(x)\rangle = \langle x, x\rangle - \langle x, T^*T(x)\rangle = \|x\|^2 - \|T(x)\|^2 = 0$.
Note: $U^* = (I - T^*T)^* = T^* - (T^*T)^* = I - T^*T = U$
i.e. U is self-adjoint. By the following lemma, $T = T_0$.
$\therefore T^*T = I$. Since T is invertible, we also have $TT^* = I$. #

Lemma: Let U be a self-adjoint operator on a finite-dim i.p.s V. If $\langle x, U(x)\rangle = 0,\ \forall x \in V$, then $U = T_0$.
Pf: Note (either $\mathbb{F} = \mathbb{C}$ or $\mathbb{F} = \mathbb{R}$) that $\exists$ an orthonormal basis β for V consisting of eigenvectors of U. Let $x \in \beta$, then $U(x) = \lambda x$ for some $\lambda \in \mathbb{F}$, and

$$0 = \langle x, U(x)\rangle = \langle x, \lambda x\rangle = \bar{\lambda}\|x\|^2 = \bar{\lambda}.$$

$\therefore \lambda = 0$
$\therefore U(x) = 0x = 0,\ \forall x \in \beta$
$U = T_0$. #