Topic#16
Adjoint of a linear operator

Goal: Recall for $A \in M_{n\times n}(\mathbb{F})$,

$$A^* \overset{def}{=} \bar{A}^t \quad (\underline{\text{conjugate transpose}} \text{ or } \underline{\text{adjoint}} \text{ of } A),$$

i.e.,

$$\langle Ax, y\rangle = \langle x, A^*y\rangle,\ \forall x, y \in \mathbb{F}^n$$

(**Exercise:** check this identity!)

How to extend to T^* for $T \in \mathcal{L}(V)$? Does it exist $T^* \in \mathcal{L}(V)$ s.t.

$$\langle Tx, y\rangle = \langle x, T^*y\rangle,\ \forall x, y \in \mathbb{F}^n?$$

<u>**Remark:**</u> If V is a **finite-dim** i.p.s. and $T \in \mathcal{L}(V)$, a natural idea is to construct T^* by $[T^*]_\beta \overset{def}{=} ([T]_\beta)^*$.

Def.: V: i.p.s. over $\mathbb{F}$ with $\langle \cdot, \cdot \rangle$ (finite-dim or ∞-dim). $T \in \mathcal{L}(V)$. Then, the **adjoint** of T, denoted by T^*, is defined to be a transformation $T^* : V \to V$ such that

$$\langle Tx, y \rangle = \langle x, T^* y \rangle, \quad \forall x, y \in V.$$

Example. For $A \in M_{n\times n}(\mathbb{F})$, $(L_A)^* = L_{A^*}$.

$$\begin{aligned}
\langle x, (L_A)^* y \rangle &= \langle L_A x, y \rangle \quad \text{(by def of } (L_A)^*\text{)} \\
&= \langle Ax, y \rangle \quad \text{(by def of } L_A\text{)} \\
&= \langle x, A^* y \rangle \quad \text{(direct computation)} \\
&= \langle x, L_{A^*} y \rangle \quad \text{(by def of } L_{A^*}\text{)}
\end{aligned}$$

$\because$ x, y are arbitrary
$\therefore$ $(L_A)^* = L_{A^*}$. □

Question: Existence? Uniqueness?

Thm. If T^* exists, then T^* is **unique** and $T^* \in \mathcal{L}(V)$.

Proof. 1°. (Uniqueness) Assume:

$$\langle T(x), y\rangle = \langle x, T^*(y)\rangle = \langle x, (T^*)'(y)\rangle, \ \forall x, y \in V.$$

then $\langle x, T^*(y) - (T^*)'(y)\rangle = 0, \ \forall x, y \in V$. Fix $y \in V$, as $x \in V$ is arbitrary,

$$T^*(y) - (T^*)'(y) = 0, \text{ i.e. } T^*(y) = (T^*)'(y).$$

As y is also arbitrary, $T^* = (T^*)'$. □

2°. (Linearity) To show

$$T^*(ax + by) = aT^*(x) + bT^*(y), \ \forall x, y \in V, \forall a, b \in \mathbb{F}.$$

In fact,
$\langle z, T^*(ax + by)\rangle = \langle T(z), ax + by\rangle = \bar{a}\langle T(z), x\rangle + \bar{b}\langle T(z), y\rangle$
$= \bar{a}\langle z, T^*(x)\rangle + \bar{b}\langle z, T^*(y)\rangle = \langle z, aT^*(x) + bT^*(y)\rangle, \ \forall z \in V.$
Therefore, $T^*(ax + by) = aT^*(x) + bT^*(y) \ \forall a, b \in \mathbb{F}, \forall x, y \in V$
. □□

Thm (existence):
If V is finite-dimensional, then T^* exists.
$(\therefore \exists! T^* \in \mathcal{L}(V) \;\; s.t \; \langle T(x), y \rangle = \langle x, T^*(y) \rangle \;\; \forall x, y \in V)$

Lemma (Riesz representaion thm):
Let V be a finite-dim i.p.s. over $\mathbb{F}$, and let $f \in \mathcal{L}(V, \mathbb{F})$.
Then, $\exists$ a unique $y \in V$ s.t. $f(x) = \langle x, y \rangle, \; \forall x \in V$

Pf of lemma. (Existence) Let $\beta = \{v_1, \cdots, v_n\}$ be an orthonormal basis for V. Let $y \overset{def}{=} \sum_{i=1}^{n} \overline{f(v_i)} v_i$, and $g(x) \overset{def}{=} \langle x, y \rangle$, $\forall x \in V$. To show $g = f$, it suffices to show $g(v_j) = f(v_j)$, $1 \leq j \leq n$. Indeed,

$$g(v_j) = \langle v_j, y \rangle = \langle v_j, \sum_{i=1}^{n} \overline{f(v_i)} v_i \rangle = \sum_{i=1}^{n} f(v_i) \langle v_j, v_i \rangle$$

$$= \sum_{i=1}^{n} f(v_i) \delta_{ji} = f(v_j).$$

$\therefore g \equiv f$ on V. □

(Uniqueness) Let $y' \in V$ be s.t. $f(x) = \langle x, y \rangle = \langle x, y' \rangle$, $\forall x \in V$. Then $\langle x, y - y' \rangle = 0$, $\forall x \in V$. $\therefore y - y' = 0_v$, i.e. $y = y'$. □

Thm. Let $T \in \mathcal{L}(V)$, where V is a finite-dim i.p.s.. Then T^* exists.

Pf: Take $y \in V$ and fix it. Def $f : V \to \mathbb{F}$ by $f(x) = \langle T(x), y\rangle$, $\forall x \in V$. It is direct to check f is linear (Exercise). Then, by lemma,

$$\exists!\ y' \in V \text{ s.t. } f(x) = \langle x, y'\rangle,\ \forall x \in V.$$

Thus, $T^* : V \to V$, $y \mapsto T^*(y) = y' \in V$ is well-defined(by previous arguments), and

$$\langle T(x), y\rangle = f(x) = \langle x, y'\rangle = \langle x, T^*(y)\rangle,$$

i.e. $\langle T(x), y\rangle = \langle x, T^*(y)\rangle$, $\forall x, y \in V$. □

Remark: Then T^* is unique & $T^* \in \mathcal{L}(V)$. □

Prop. Let $T \in \mathcal{L}(V)$, where V is a finite-dim i.p.s with an orthonormal o.b. β. Then

$$[T^*]_\beta = [T]_\beta^*.$$

Pf. Let $\beta = \{v_1, \cdots, v_n\}$, and $[T]_\beta = A$, $[T^*]_\beta = B$. Then,

$$\begin{aligned} B_{ij} &= \langle T^*(v_j), v_i \rangle \\ &= \langle v_j, T(v_i) \rangle \\ &= \overline{\langle T(v_i), v_j \rangle} \\ &= \overline{A_{ji}} \end{aligned}$$

i.e. $B = A^*$. □

Remarks:

1°. This gives an alternative way to construct T^* explicitly in terms of $([T]_\beta)^*$.

Properties: Let $T, U \in \mathcal{L}(V)$, where V is an i.p.s. (finite-dim or ∞-dim). Assume $T^*, U^* \in \mathcal{L}(V)$ exist. Then
(a) $(T+U)^* = T^* + U^*$.
(b) $(cT)^* = \bar{c}T^*$, $\forall c \in \mathbb{F}$.
(c) $(TU)^* = U^*T^*$.
(d) $T^{**} = T$.
(e) $I^* = I$.

Remark: Similar properties are true for $n \times n$ matrices, i.e., let $A, B \in M_{n\times n}(\mathbb{F})$, then
$(A+B)^* = A^* + B^*$,
$(cA)^* = \bar{c}A^*$,
$(AB)^* = B^*A^*$,
$A^{**} = A$,
$I_n^* = I_n$. □

Proof of (b), (c), (e) left for exercises.

Proof of (a).

$$\begin{aligned}\langle x, (T+U)^*(y)\rangle &= \langle (T+U)(x), y\rangle \\ &= \langle T(x)+U(x), y\rangle \\ &= \langle T(x), y\rangle + \langle U(x), y\rangle \\ &= \langle x, T^*(y)\rangle + \langle x, U^*(y)\rangle \\ &= \langle x, T^*(y)+U^*(y)\rangle \\ &= \langle x, (T^*+U^*)(y)\rangle.\end{aligned}$$

$\because x, y$ are arbitrary
$\therefore (T+U)^* = T^* + U^*$. □

Proof of (d):

$$\langle x, T^{**}(y)\rangle = \langle T^*(x), y\rangle = \langle x, T(y)\rangle.$$

$\therefore T^{**} = T$. □□