

# **Topic#9**

## **Change of coordinates**

Let  $V$ : v.s.,  $\dim(V) = n < \infty$ ,  $\beta, \beta'$ : o.b. for  $V$ .

$$\begin{array}{ccc}
 v \in V & \xrightarrow{I_V \in \mathcal{L}(V), \text{invertible}} & v \in V \\
 \downarrow [\cdot]_{\beta'} & & \downarrow [\cdot]_{\beta} \\
 [v]_{\beta'} \in \mathbb{F}^n & \xrightarrow{Q \stackrel{\text{def}}{=} [I_V]_{\beta'}^{\beta} \in M_{n \times n}(\mathbb{F})} & [v]_{\beta} \in \mathbb{F}^n
 \end{array}$$

Then,

$$[v]_{\beta} = [I_V]_{\beta'}^{\beta} [v]_{\beta'} = Q [v]_{\beta'} \text{ where } Q \stackrel{\text{def}}{=} [I_V]_{\beta'}^{\beta}$$

$Q = [I_V]_{\beta'}^{\beta}$  is called a change of coordinate matrix.

It changes  $\beta'$ -coordinate of  $v$  to  $\beta$ -coordinate of  $v$ .

## Notes:

$$1^\circ. Q^{-1} = ([I_V]_{\beta'}^\beta)^{-1} = [I_V^{-1}]_{\beta}^{\beta'} = [I_V]_{\beta}^{\beta'}$$

$$2^\circ. \beta' = \{v'_1, \dots, v'_n\}, \beta = \{v_1, \dots, v_n\}$$

$$Q = [I_V]_{\beta'}^\beta = ([I_V(v'_1)]_{\beta}, \dots, [I_V(v'_n)]_{\beta}) = ([v'_1]_{\beta}, \dots, [v'_n]_{\beta})$$

**Prop:**  $V \xrightarrow{T \in \mathcal{L}(V)} V$  where  $\dim(V) = n < \infty$  and  $\beta, \beta' : \text{o.b. for } V$ .

Relation of  $[T]_{\beta}$  and  $[T]_{\beta'}$  ?

$$[T]_{\beta'} = [T]_{\beta'}^{\beta'} = [I_V \circ T \circ I_V]_{\beta'}^{\beta'} = [I_V]_{\beta}^{\beta'} \cdot [T]_{\beta}^{\beta} \cdot [I_V]_{\beta'}^{\beta} = Q^{-1} [T]_{\beta}^{\beta} Q$$

Then,  $[T]_{\beta'} = Q^{-1} [T]_{\beta} Q$ , where  $Q = [I_V]_{\beta'}^{\beta}$ .

## Remark:

$$A \in M_{n \times n}(\mathbb{F})$$

1°. Let  $\gamma' = \{e_1, \dots, e_n\}$  be s.o.b. for  $\mathbb{F}^n$ .

$$[L_A]_{\gamma'} = ([L_A(e_1)]_{\gamma'}, \dots, [L_A(e_n)]_{\gamma'}) = ([Ae_1]_{\gamma'}, \dots, [Ae_n]_{\gamma'}) = A.$$

$Ae_1$ : 1<sup>st</sup> column of  $A$ ,  $\gamma'$ : s.o.b.  $\Rightarrow [Ae_1]_{\gamma'}$  is still 1<sup>st</sup> column of  $A$ .

2° Let  $\gamma = \{v_1, \dots, v_n\}$  be an o.b. for  $\mathbb{F}^n$ .

$$[L_A]_{\gamma} = [L_A]_{\gamma}^{\gamma'} = [I_{\mathbb{F}^n} \circ L_A \circ I_{\mathbb{F}^n}]_{\gamma}^{\gamma'} = [I_{\mathbb{F}^n}]_{\gamma}^{\gamma'} [L_A]_{\gamma'}^{\gamma'} [I_{\mathbb{F}^n}]_{\gamma'}^{\gamma'} = Q^{-1}AQ$$

$\gamma'$ : s.o.b. for  $\mathbb{F}^n$

$$\text{And } Q \stackrel{\text{def}}{=} [I_{\mathbb{F}^n}]_{\gamma}^{\gamma'} = ([v_1]_{\gamma'} \mid [v_2]_{\gamma'} \mid \dots \mid [v_n]_{\gamma'}) = (v_1 \mid v_2 \mid \dots \mid v_n)$$

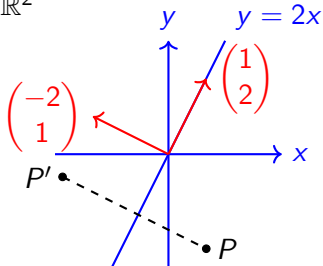
$$\text{i.e. } Q = (v_1 \mid v_2 \mid \dots \mid v_n)$$

$$\therefore [L_A]_{\gamma=\{v_1, \dots, v_n\}} = Q^{-1}AQ$$

$$\text{where } Q = [I_v]_{\gamma}^{\gamma'} = (v_1 \mid v_2 \mid \dots \mid v_n)_{n \times n}.$$

e.g.

$\mathbb{R}^2$



$T \stackrel{\text{def}}{=} \text{the reflection}$   
about the line  $y = 2x$

**Question:** Find  $[T]_{\beta}$ , where  $\beta$   
is the s.o.b. for  $\mathbb{R}^2$ .

**Sln:**  $\beta' = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \end{pmatrix} \right\}$  is another o.b. for  $\mathbb{R}^2$ . By def. of  $T$ ,

$$T \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + 0 \begin{pmatrix} -2 \\ 1 \end{pmatrix},$$
$$T \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} = 0 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + (-1) \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

$$\therefore [T]_{\beta'} = [T]_{\beta'}^{\beta'} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let  $Q = [I_{\mathbb{R}^2}]_{\beta'}^{\beta} = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$  (change  $\beta'$ -co to  $\beta$ -co).

$$\therefore Q^{-1} = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}^{-1} = \frac{1}{5} \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \text{ (change } \beta\text{-co to } \beta'\text{-co)}.$$

Therefore,

$$\begin{aligned} [T]_{\beta} &= [I_{\mathbb{R}^2} \circ T \circ I_{\mathbb{R}^2}]_{\beta}^{\beta} \\ &= [I_{\mathbb{R}^2}]_{\beta'}^{\beta} [T]_{\beta'}^{\beta'} [I_{\mathbb{R}^2}]_{\beta}^{\beta'} \\ &= Q [T]_{\beta'}^{\beta'} Q^{-1} \\ &= \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \frac{1}{5} \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \\ &= \frac{1}{5} \begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix} := A. \end{aligned}$$

Since  $\beta$  is a s.o.b. for  $\mathbb{R}^2$ , we have  $T = L_A$ . □