Plan for Chapter 2: (Five topics in total)
Topic #5 Linear transformations
Topic #6 Null space, range, and dimension theorem
Topic #7 Matrix representation of a linear transformation
Topic #8 Invertibility and isomorphism
Topic #9 Change of coordinates

Topic#5
Linear transformations

Def.: Let V, W be v.s. over $\mathbb{F}$. A function $T : V \to W$ is linear if $\forall x, y \in V, \forall a \in \mathbb{F}$,

(1) $T(x + y) = T(x) + T(y)$, and
(2) $T(ax) = aT(x)$.

$T : V \to W$ is called a **linear transformation** from V to W if the function $T : V \to W$ is linear.

Notation:

$$\mathcal{L}(V, W) \stackrel{def}{=} \text{set of all linear transformations from } V \text{ to } W.$$

$$\mathcal{L}(V) \stackrel{def}{=} \mathcal{L}(V, V) \text{ in case when } W = V$$

Quick Consequences:

(1) If $T : V \to W$ is linear, then $T(0_V) = 0_W$.

Pf.: $T(0_V) = T(00_V) = 0T(0_V) = 0_W$.

(2) $T : V \to W$ is linear **IFF**

$$T(ax + y) = aT(x) + T(y), \forall x, y \in V, \forall a \in \mathbb{F},$$

IFF

$$T(\textstyle\sum_{i=1}^{n} a_i x_i) = \sum_{i=1}^{n} a_i T(x_i),$$
$$\forall x_1, \cdots, x_n \in V, \forall a_1, \cdots, a_n \in \mathbb{F}. \ (n \geq 2)$$

(T preserves the linear combination)

A linear transformation over a finite-dimensional v.s. is completely determined by its action on a basis.

Thm.: Let V, W be vector space over $\mathbb{F}$. Assume that V is finite-dimensional with a basis $\{v_1, \cdots, v_n\}$. Then, $\forall w_1, \cdots, w_n \in W$, $\exists!$ linear transformation $T : V \to W$ s.t. $T(v_i) = w_i, i = 1, \cdots, n$.

Proof. (Existence)

Let $v \in V = \text{span}(\beta) = \text{span}(\{v_1, \cdots, v_n\})$. Then, $\exists! a_1, \cdots, a_n \in \mathbb{F}$ s.t. $v = \sum_{i=1}^{n} a_i v_i$.
Define

$$T(v) \overset{def}{=} \sum_{i=1}^{n} a_i w_i.$$

Then, $T : V \to W$ is well-defined.

Claim: $T : V \to W$ is linear. Proof: to show $\forall u, v \in V$, $\forall a \in \mathbb{F}$,

$$(1)\ T(u+v) = T(u) + T(v), \text{ and } (2)\ T(au) = aT(u).$$

Let

$$u = \sum_{i=1}^{n} b_i v_i \in V, \quad v = \sum_{i=1}^{n} c_i v_i \in V,$$

then

$$T(u) = \sum_{i=1}^{n} b_i w_i, \quad T(v) = \sum_{i=1}^{n} c_i w_i.$$

Proof of (1): Note

$$u + v = \sum_{i=1}^{n} b_i v_i + \sum_{i=1}^{n} c_i v_i = \sum_{i=1}^{n} (b_i + c_i) v_i$$

By def of T,

$$\begin{aligned} T(u+v) &= \sum_{i=1}^{n} (b_i + c_i) w_i \\ &= \sum_{i=1}^{n} b_i w_i + \sum_{i=1}^{n} c_i w_i \quad (both \in W) \\ &= T(u) + T(v). \end{aligned}$$

Proof of (2): Note $au = \sum_{i=1}^{n} (ab_i) v_i \in V$. Then, by def of T,

$$T(au) = T(\sum_{i=1}^{n} ab_i v_i) = \sum_{i=1}^{n} (ab_i) w_i = a \sum_{i=1}^{n} b_i w_i = aT(u).$$

□

(Uniqueness)

Assume that $\tilde{T} : V \to W$ is also linear such that

$$\tilde{T}(v_i) = w_i \quad (i = 1, \cdots, n),$$

to show: $\tilde{T} = T$, i.e. $\tilde{T}(v) = T(v)$, $\forall v \in V$.

Take $v \in V =$span(β), $v = \sum_{i=1}^{n} a_i v_i \in V$. Then

$$\begin{aligned} \tilde{T}(v) &= \tilde{T}(\sum_{i=1}^{n} a_i v_i) \\ &= \sum_{i=1}^{n} a_i \tilde{T}(v_i) \quad (\tilde{T} \text{ is linear}) \\ &= \sum_{i=1}^{n} a_i w_i \quad (\tilde{T}(v_i) = w_i, i = 1, \cdots, n) \\ &\overset{\text{existence proof}}{=} T(v) \quad (\text{def. of } T). \end{aligned}$$

□□

Examples:

(1) $A_{m \times n} \in M_{m \times n}(\mathbb{F})$.

$\mathbb{F}^n, \mathbb{F}^m$: understood as v.s. of column vectors

Then the function

$$\forall x \in \mathbb{F}^n \mapsto L_A(x) \stackrel{def}{=} Ax \in \mathbb{F}^m$$

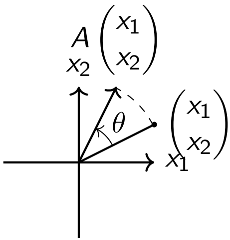
$$\text{Then } (Ax)_i = \sum_{j=1}^{n} a_{ij} x_j, 1 \leq i \leq m$$

Verify $L_A \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$.

L_A is called a **left-multiplication transformation**.

For instance

- $A \stackrel{def}{=} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.



$$
\begin{aligned}
x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \mapsto L_A(x) &= Ax \\
&= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\
&= \begin{pmatrix} x_1\cos\theta - x_2\sin\theta \\ x_1\sin\theta + x_2\cos\theta \end{pmatrix} \in \mathbb{R}^2
\end{aligned}
$$

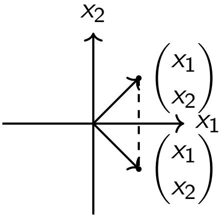
L_A is also called a **rotation** by θ in the anti-clockwise direction. □

• $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \mapsto L_A(X) =$

$Ax = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ -x_2 \end{pmatrix}$

L_A is also called the **reflection** about the x_1-axis. #

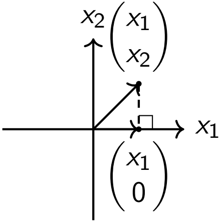


• $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \mapsto L_A(X) =$

$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}$

L_A is also called the **projection** on the x_1-axis. □



(2) For any v.s. V and W over $\mathbb{F}$.

$T_0 : V \to W, \forall x \in V \mapsto T_0(x) \overset{def}{=} 0_W$: the zero transformation.

$I_V : V \to V, \forall x \in V \mapsto I_V(x) \overset{def}{=} x$: the identity transformation.

(Exercise: $T_0 \in \mathcal{L}(V, W)$, $I_V \in \mathcal{L}(V)$)

(3):

(a) T: $M_{m\times n}(\mathbb{F}) \mapsto M_{m\times n}(\mathbb{F}) \quad A \mapsto T(A) \overset{def}{=} A^t$

defines a linear transformation from $M_{m\times n}(\mathbb{F})$ to $M_{m\times n}(\mathbb{F})$.

(b) T: $f \in P_n(\mathbb{R}) \mapsto T(f) \in P_{n-1}(\mathbb{R})$ given by

$$T(f)(x) = f'(x), \forall x \in \mathbb{R},$$

defines a linear transformation from $P_n(\mathbb{R})$ to $P_{n-1}(\mathbb{R})$..

(c)

$$T : C(\mathbb{R}) \mapsto R,\ f \mapsto T(f) \overset{def}{=} \int_a^b f(t)dt \in \mathbb{R}, (-\infty < a < b < \infty)$$

defines a linear transformation from $C(\mathbb{R})$ to $\mathbb{R}$.