MATH2040A Homework 5
Reference Solutions

1 Compulsory Part

2.4.14. Let

$$V = \left\{ \begin{pmatrix} a & a+b \\ 0 & c \end{pmatrix} : a, b, c \in \mathbb{F} \right\}$$

Construct an isomorphism from V to $\mathbb{F}^3$.

Solution: Define $T : V \to \mathbb{F}^3$ by $T\begin{pmatrix} a & a+b \\ 0 & c \end{pmatrix} = (a, b, c)$ for $\begin{pmatrix} a & a+b \\ 0 & c \end{pmatrix} \in V$. It is easy to see that the map is well-defined.

Let $A_1 = \begin{pmatrix} a_1 & a_1+b_1 \\ 0 & c_1 \end{pmatrix}, A_2 = \begin{pmatrix} a_2 & a_2+b_2 \\ 0 & c_2 \end{pmatrix} \in V$ and $\lambda \in \mathbb{F}$. Then $T(\lambda A_1 + A_2) = T\begin{pmatrix} \lambda a_1 + a_2 & \lambda(a_1+b_1) + (a_2+b_2) \\ 0 & \lambda c_1 + c_2 \end{pmatrix} = \begin{pmatrix} \lambda a_1 + a_2 & (\lambda a_1 + a_2) + (\lambda b_1 + b_2) \\ 0 & \lambda c_1 + c_2 \end{pmatrix} = (\lambda a_1 + a_2, \lambda b_1 + b_2, \lambda c_1 + c_2) = \lambda(a_1, b_1, c_1) + (a_2, b_2, c_2) = \lambda T(A_1) + T(A_2)$. Since A_1, A_2, λ are arbitrary, T is linear.

Let $A \in V$ be such that $T(A) = 0 = (0, 0, 0)$. Then with $a, b, c \in \mathbb{F}$ such that $A = \begin{pmatrix} a & a+b \\ 0 & c \end{pmatrix}$, we have $T(A) = T\begin{pmatrix} a & a+b \\ 0 & c \end{pmatrix} = (a, b, c) = (0, 0, 0)$, so $A = \begin{pmatrix} a & a+b \\ 0 & c \end{pmatrix} = 0$. Since A is arbitrary, $\mathsf{N}(T) = \{0\}$.

Let $v = (a, b, c) \in \mathbb{F}^3$. Let $A = \begin{pmatrix} a & a+b \\ 0 & c \end{pmatrix} \in V$. Then $T(A) = T\begin{pmatrix} a & a+b \\ 0 & c \end{pmatrix} = (a, b, c) = v$ by definition. This implies that $v \in \mathsf{R}(T)$. As v is arbitrary, $\mathsf{R}(T) = \mathbb{F}^3$.

So T is a linear transformation that is one-to-one and onto, hence an isomorphism from V to $\mathbb{F}^3$.

2.4.15. Let V and W be n-dimensional vector spaces, and let $T : V \to W$ be a linear transformation. Suppose that β is a basis for V. Prove that T is an isomorphism if and only if $T(\beta)$ is a basis of W.

Solution:

(a) Suppose T is an isomorphism. Then T is one-to-one and onto. By Question 2.1.14(c) in HW3, $T(\beta)$ is a basis of W.

(b) Suppose $T(\beta)$ is a basis of W. It suffices to show that T is one-to-one and onto. We may assume that $\beta = \{v_1, \ldots, v_n\}$.

- Let $v \in V$ such that $Tv = 0$. Then there exists scalars $a_1, \ldots, a_n$ such that $v = \sum_{i=1}^n a_i v_i$, so $0 = Tv = T(\sum_{i=1}^n a_i v_i) = \sum_{i=1}^n a_i T v_i$. Since $T(\beta) = \{T(v_1), \ldots, T(v_n)\}$ is a basis, it is linearly independent and so $a_1 = \ldots = a_n = 0$. This implies that $v = \sum_{i=1}^n a_i v_i = 0$. As v is arbitrary, $\mathsf{N}(T) = \{0\}$.
- Let $w \in W$. Then there exists scalars $a_1, \ldots, a_n$ such that $w = \sum_{i=1}^n a_i T(v_i) = T(\sum_{i=1}^n a_i v_i) \in \mathsf{R}(T)$. As w is arbitrary, $\mathsf{R}(T) = W$.

Hence T is both one-to-one and onto, and so is an isomorphism.

2.4.16. Let B be an $n \times n$ invertible matrix. Define $\Phi : M_{n\times n}(\mathbb{F}) \to M_{n\times n}(\mathbb{F})$ by $\Phi(A) = B^{-1}AB$. Prove that Φ is an isomorphism.

Solution:

- We first show that Φ is linear. For all $C, D \in M_{n\times n}(\mathbb{F})$ and $\alpha \in \mathbb{F}$, we have $\Phi(\alpha C + D) = B^{-1}(\alpha C + D)B = B^{-1}(\alpha C)B + B^{-1}DB = \alpha B^{-1}CD + B^{-1}DB = \alpha\Phi(C) + \Phi(D)$. So Φ is linear.

- Let $A \in M_{n\times n}(\mathbb{F})$ be such that $\Phi(A) = 0$. Then $B^{-1}AB = 0$, $A = B(B^{-1}AB)B^{-1} = B0B^{-1} = 0$. As A is arbitrary, $\mathsf{N}(\Phi) = \{0\}$.
- Let $A \in M_{n\times n}(\mathbb{F})$. Then $BAB^{-1} \in M_{n\times n}(\mathbb{F})$, and $\Phi(BAB^{-1}) = B^{-1}(BAB^{-1})B = A \in \mathsf{R}(\Phi)$. As A is arbitrary, $\mathsf{R}(\Phi) = M_{n\times n}(\mathbb{F})$.

Hence Φ is a linear transformation that is one-to-one and onto, and so is an isomorphism.

Note

If we denote $\Psi_C(A) = C^{-1}AC$ for invertible C, then $\Phi^{-1} = (\Psi_B)^{-1} = \Psi_{B^{-1}}$. Also for the students who know some group theory, note that the (linear) isomorphism concerns only its linear structure, not its (multiplicative) group structure.

2.5.2(d). For the following pair of order bases β and β' for $\mathbb{R}^2$, find the change of coordinate matrix that changes β'-coordinates into β-coordinates.

$\beta = \{(-4,3),(2,-1)\}$ and $\beta' = \{(2,1),(-4,1)\}$

Solution: As $(2,1) = 2\cdot(-4,3) + 5\cdot(2,-1)$ and $(-4,1) = -1(-4,3) - 4\cdot(2,-1)$, we have $[\mathrm{Id}]_{\beta'}^{\beta} = \begin{pmatrix} 2 & -1 \\ 5 & -4 \end{pmatrix}$

Note

Alternatively, $[\mathrm{Id}]_{\beta'}^{\beta} = [\mathrm{Id}]_{\alpha}^{\beta}[\mathrm{Id}]_{\beta'}^{\alpha} = ([\mathrm{Id}]_{\beta}^{\alpha})^{-1}[\mathrm{Id}]_{\beta'}^{\alpha} = \begin{pmatrix} -4 & 2 \\ 3 & -1 \end{pmatrix}^{-1}\begin{pmatrix} 2 & -4 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 5 & -4 \end{pmatrix}$.

There are many way to handle questions like this one. Solving for the coefficients of the linear combinations is just one of them.

2.5.3(f). For the following pair of ordered bases β and β' for $\mathsf{P}_2(\mathbb{R})$, find the change of coordinate matrix that changes β'-coordinates into β-coordinates.

$\beta = \{2x^2 - x + 1, x^2 + 3x - 2, -x^2 + 2x + 1\}$ and $\beta' = \{9x - 9, x^2 + 21x - 2, 3x^2 + 5x + 2\}$

Solution: Let $\alpha = \{1, x, x^2\}$ be the standard ordered basis of $\mathsf{P}_2(\mathbb{R})$. Then $[\mathrm{Id}]_{\beta}^{\alpha} = \begin{pmatrix} 1 & -2 & 1 \\ -1 & 3 & 2 \\ 2 & 1 & -1 \end{pmatrix}$ and $[\mathrm{Id}]_{\beta'}^{\alpha} = \begin{pmatrix} -9 & -2 & 2 \\ 9 & 21 & 5 \\ 0 & 1 & 3 \end{pmatrix}$. Since $[\mathrm{Id}]_{\beta}^{\alpha}[\mathrm{Id}]_{\beta'}^{\beta} = [\mathrm{Id}]_{\alpha}^{\beta}$, we can solve for $[\mathrm{Id}]_{\beta'}^{\beta}$ (e.g. by elimination) and get $[\mathrm{Id}]_{\beta'}^{\beta} = ([\mathrm{Id}]_{\beta}^{\alpha})^{-1}[\mathrm{Id}]_{\alpha}^{\beta} = \begin{pmatrix} -2 & 1 & 2 \\ 3 & 4 & 1 \\ -1 & 5 & 2 \end{pmatrix}$

2.5.4. Let T be the linear operator on $\mathbb{R}^2$ defined by $T\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2a+b \\ a-3b \end{pmatrix}$, let β be the standard ordered basis for $\mathbb{R}^2$, and let $\beta' = \left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right\}$. Use Theorem 2.23 and the fact that $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$ to find $[T]_{\beta'}$.

Solution: Since $T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 2\cdot\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - 3\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, we have $[T]_{\beta} = \begin{pmatrix} 2 & 1 \\ 1 & -3 \end{pmatrix}$.

Since $\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2\cdot\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, the change of coordinate matrix is $[\mathrm{Id}]_{\beta'}^{\beta} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$.

Thus

$$[T]_{\beta'} = [\mathrm{Id}]_{\beta}^{\beta'}[T]_{\beta}[\mathrm{Id}]_{\beta'}^{\beta} = [\mathrm{Id}]_{\beta'}^{\beta\ -1}[T]_{\beta}[\mathrm{Id}]_{\beta'}^{\beta} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^{-1}\begin{pmatrix} 2 & 1 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 8 & 13 \\ -5 & -9 \end{pmatrix}$$

2.5.6(d). For the matrix A and ordered basis β, find $[\mathsf{L}_A]_\beta$. Also, find an invertible matrix Q such that $[\mathsf{L}_A]_\beta = Q^{-1}AQ$.

$$A = \begin{pmatrix} 13 & 1 & 4 \\ 1 & 13 & 4 \\ 4 & 4 & 10 \end{pmatrix} \text{ and } \beta = \left\{ \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

Solution: By computation, we have

$$\mathsf{L}_A \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 6 \\ 6 \\ -12 \end{pmatrix} = 6 \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$
$$\mathsf{L}_A \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 12 \\ -12 \\ 0 \end{pmatrix} = 12 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$
$$\mathsf{L}_A \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 18 \\ 18 \\ 18 \end{pmatrix} = 18 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

So $[\mathsf{L}_A]_\beta = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 18 \end{pmatrix}$.

Let α be the standard ordered basis for $\mathbb{F}^3$. Then by Theorem 2.15, $A = [\mathsf{L}_A]_\alpha$. So $[\mathsf{L}_A]_\beta = Q^{-1}[\mathsf{L}_A]_\alpha Q$ with $Q = [\mathrm{Id}]_\beta^\alpha = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ -2 & 0 & 1 \end{pmatrix}$.

Note

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ -2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 6 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 18 \end{pmatrix} = \begin{pmatrix} 13 & 1 & 4 \\ 1 & 13 & 4 \\ 4 & 4 & 10 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ -2 & 0 & 1 \end{pmatrix}$$

2.5.7. In $\mathbb{R}^2$, let L be the line $y = mx$, where $m \neq 0$. Find an expression for $T(x, y)$, where

(a) T is the reflection of $\mathbb{R}^2$ about L

(b) T is the projection on L along the line perpendicular to L

Solution:

(a) By high school mathematics, the reflected point (a, b) should satisfy that $\left(\frac{a+x}{2}, \frac{b+y}{2}\right) \in L$ and $(a - x, b - y) \in L^\perp$ with $L^\perp$ being the line that passes through origin and is perpendicular to L, hence with slope $-1/m$. This implies that $\frac{b+y}{2} = m\frac{a+x}{2}$ and $b - y = -\frac{1}{m}(a - x)$. Solving the system gives $(a, b) = \left(\frac{(1-m^2)x+2my}{1+m^2}, \frac{2mx-(1-m^2)y}{1+m^2}\right)$, so $T(x, y) = \left(\frac{(1-m^2)x+2my}{1+m^2}, \frac{2mx-(1-m^2)y}{1+m^2}\right)$

(b) By high school mathematics, the projected point (a, b) should satisfy $(a, b) \in L$ and $(x - y) - (a, b) \in L^\perp$. This implies that $b = ma$ and $y - b = -\frac{1}{m}(x - a)$. Solving the system gives $(a, b) = \left(\frac{x+my}{1+m^2}, \frac{mx+m^2y}{1+m^2}\right)$. So $T(x, y) = \left(\frac{x+my}{1+m^2}, \frac{mx+m^2y}{1+m^2}\right)$

Alternatively,

(a) It is easy to see that $T\begin{pmatrix} 1 \\ m \end{pmatrix} = \begin{pmatrix} 1 \\ m \end{pmatrix}$ and $T\begin{pmatrix} -m \\ 1 \end{pmatrix} = \begin{pmatrix} m \\ -1 \end{pmatrix}$. Let $\beta = \left\{ \begin{pmatrix} 1 \\ m \end{pmatrix}, \begin{pmatrix} -m \\ 1 \end{pmatrix} \right\}$. It is easy to verify that β is a basis of $\mathbb{R}^2$. Since T is linear, $[T]_\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Hence in the standard ordered basis $\alpha = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$, $[T]_\alpha =$ $[\mathrm{Id}]_\beta^\alpha [T]_\beta [\mathrm{Id}]_\alpha^\beta = \begin{pmatrix} 1 & -m \\ m & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & -m \\ m & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1-m^2}{1+m^2} & \frac{2m}{1+m^2} \\ \frac{2m}{1+m^2} & \frac{m^2-1}{1+m^2} \end{pmatrix}$, so $T(x, y) = \left(\frac{(1-m^2)x+2my}{1+m^2}, \frac{2mx-(1-m^2)y}{1+m^2}\right)$

(b) It is easy to see that $T\begin{pmatrix} 1 \\ m \end{pmatrix} = \begin{pmatrix} 1 \\ m \end{pmatrix}$ and $T\begin{pmatrix} -m \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Let β be defined as in the previous part. As T is linear, $[T]_\beta = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Hence in the standard ordered basis α, $[T]_\alpha = \begin{pmatrix} 1 & -m \\ m & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -m \\ m & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{1+m^2} & \frac{m}{1+m^2} \\ \frac{m}{1+m^2} & \frac{m^2}{1+m^2} \end{pmatrix}$, so $T(x,y) = \left(\frac{x+my}{1+m^2}, \frac{mx+m^2y}{1+m^2}\right)$

2 Optional Part

2.4.1. Label the following statements as true or false. In each part, V and W are vector spaces with ordered (finite) bases α and β, respectivelt, $T : V \to W$ is linear, and A and B are matrices.

(a) $\left([T]_\alpha^\beta\right)^{-1} = [T^{-1}]_\alpha^\beta$

(b) T is invertible if and only if T is one-to-one and onto

(c) $T = \mathsf{L}_A$ where $A = [T]_\alpha^\beta$

(d) $M_{2\times 2}(\mathbb{F})$ is isomorphic to $\mathbb{F}^5$

(e) $\mathsf{P}_n(\mathbb{F})$ is isomorphic to $\mathsf{P}_n(\mathbb{F})$ if and only if $n = m$

(f) $AB = I$ implies that A and B are invertible

(g) If A is invertible, then $\left(A^{-1}\right)^{-1} = A$

(h) A is invertible if and only if L_A is invertible

(i) A must be square in ordered to process an inverse

Solution:

a) False

b) True

c) False. Consider $V \neq \mathbb{F}^{|\alpha|}$. See also Theorem 2.15

d) False

e) True

f) False. Consider $A = \begin{pmatrix} 0 & 1 \end{pmatrix} \in M_{1\times 2}(\mathbb{R})$, $B = \begin{pmatrix} 0 & 1 \end{pmatrix}^\mathsf{T} \in M_{2\times 1}(\mathbb{R})$

g) True

h) True

i) True

2.4.22. Let $c_0, \dots, c_n$ be distinct scalars from an infinite field $\mathbb{F}$. Define $T : \mathsf{P}_n(\mathbb{F}) \to \mathbb{F}^{n+1}$ by $T(f) = (f(c_0), \dots, f(c_n))$. Prove that T is an isomorphism.

Solution: We first show that T is linear. For all $f, g \in \mathsf{P}_n(\mathbb{F})$ and $\alpha \in \mathbb{F}$, $T(\alpha f + g) = ((\alpha f + g)(c_0), \dots, (\alpha f + g)(c_n)) = (\alpha f(c_0) + g(c_0), \dots, \alpha f(c_n) + g(c_n)) = \alpha(f(c_0), \dots, f(c_n)) + (g(c_0), \dots, g(c_n)) = \alpha T(f) + T(g)$. So T is linear.

Let $f \in \mathsf{P}_n(\mathbb{F})$ be such that $T(f) = 0$. Then $(f(c_0), \dots, f(c_n)) = (0, \dots, 0)$, so $f(c_i) = 0$ for all $i \in \{0, \dots, n\}$. Since $c_0, \dots, c_n$ are distinct, by factor theorem we have $f(x) = g(x)(x - c_0)\dots(x - c_n)$ for some $g \in \mathsf{P}(\mathbb{F})$. Then $n \geq \deg f(x) = \deg\left(g(x)(x - c_0)\dots(x - c_n)\right) = n + 1 + \deg g$, so $\deg g < 0$. This implies that $g = 0$ and so $f(x) = g(x)(x - c_0)\dots(x - c_n) = 0$. As f is arbitrary, $\mathsf{N}(T) = \{0\}$.

Let $(d_0, \dots, d_n) \in \mathbb{F}^{n+1}$. Define $f(x) = \sum\limits_{i=0}^{n} \left(d_i \prod\limits_{\substack{j \in \{0,\dots,n\} \\ j \neq i}} \frac{1}{c_i - c_j} \right) \prod\limits_{\substack{j \in \{0,\dots,n\} \\ j \neq i}} (x - c_j) \in \mathsf{P}(\mathbb{F})$. Then $\deg f \leq n$ and so $f \in \mathsf{P}_n(\mathbb{F})$.

Also for all $k \in \{0, \dots, n\}$, $f(c_k) = \sum\limits_{i=0}^{n} \left(d_i \prod\limits_{\substack{j \in \{0,\dots,n\} \\ j \neq i}} \frac{1}{c_i - c_j} \right) \prod\limits_{\substack{j \in \{0,\dots,n\} \\ j \neq i}} (c_k - c_j) = \sum\limits_{i=0}^{n} d_i \prod\limits_{\substack{j \in \{0,\dots,n\} \\ j \neq i}} \frac{c_k - c_j}{c_i - c_j} = \sum_{i=0}^n d_i \delta_{ik} = d_k$ with $\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$, so $T(f) = (f(c_0), \dots, f(c_n)) = (d_0, \dots, d_n)$. This implies that $(d_0, \dots, d_n) \in \mathsf{R}(T)$. As $(d_0, \dots, d_n)$ is arbitrary, $\mathsf{R}(T) = \mathbb{F}^{n+1}$.

Hence T is a linear transformation that is one-to-one and onto, and so an isomorphism.

Note

It seems that the assumption of the field being infinite can be removed. See also Shamir's Secret Sharing Scheme (SSSS).

2.4.23. Let V denote the vector space of $\mathbb{F}$-sequences that have only a finite number of nonzero terms, and $W = \mathsf{P}(\mathbb{F})$. Define $T : V \to W$ by $T(\sigma) = \sum_{i=0}^n \sigma(i)x^i$, where n is the largest integer such that $\sigma(n) \neq 0$. Prove that T is an isomorphism.

Solution: We first show that T is linear. Let $\sigma, \tau \in V$ and $c \in \mathbb{F}$. Let $n_\sigma, n_\tau, n_{c\sigma+\tau} \in \mathbb{N}$ be the largest integers such that $\sigma(n_\sigma) \neq 0$, $\tau(n_\tau) \neq 0$, and $(c\sigma+\tau)(n_{c\sigma+\tau}) \neq 0$. Then for $n > \max(n_\sigma, n_\tau)$, $\sigma(n) = \tau(n) = 0$, so $(c\sigma+\tau)(n) = c0+0 = 0$. This implies that $n_{c\sigma+\tau} \leq \max(n_\sigma, n_\tau)$. So $T(c\sigma+\tau) = \sum_{i=0}^{n_{c\sigma+\tau}} (c\sigma+\tau)(i)x^i = \sum_{i=0}^{\max(n_\sigma,n_\tau)} (c\sigma+\tau)(i)x^i = c\sum_{i=0}^{\max(n_\sigma,n_\tau)} \sigma(i)x^i + \sum_{i=0}^{\max(n_\sigma,n_\tau)} \tau(i)x^i = c\sum_{i=0}^{n_\sigma} \sigma(i)x^i + \sum_{i=0}^{n_\tau} \tau(i)x^i = cT(\sigma) + T(\tau)$. Since σ, τ, c are arbitrary, T is linear.

Let $\sigma \in V$ be such that $T(\sigma) = 0$. Then with n being the largest integer such that $\sigma(n) \neq 0$, we have $0 = T(\sigma) = \sum_{i=0}^{n} \sigma(i)x^i$, so $\sigma(i) = 0$ for all $i \in \{0, \dots, n\}$. So by definition of n, we must have $n < 0$. This implies that $\sigma = 0$ is the zero sequence. As σ is arbitrary, $\mathsf{N}(T) = \{0\}$.

Let $f \in W = \mathsf{P}(\mathbb{F})$. Then there exists $n \in \mathbb{Z}$ such that $f(x) = \sum_{i=0}^{n} a_i x^i$ for some scalars $a_0, \dots, a_n$ (with $n < 0$ indicating $f = 0$). Let $\sigma = (c_0, c_1, \dots)$ be the sequence such that $c_i = a_i$ for $i \leq n$ and $c_j = 0$ for $j > n$. As n is finite, $\sigma \in V$ and $\sigma(k) = c_k = 0$ for $k > n$. By definition, we have $T(\sigma) = \sum_{i=0}^{n} \sigma(i)x^i = \sum_{i=0}^{n} a_i x^i = f(x)$. This implies that $f \in \mathsf{R}(T)$. As f is arbitrary, we have $\mathsf{R}(T) = W$.

Hence T is a linear transformation that is one-to-one and onto, and so is an isomorphism.

Note

Compare this with the previous question (Question 2.4.22).

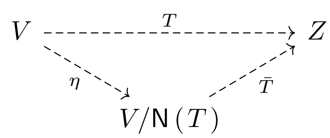
2.4.24. Let $T : V \to Z$ be a linear transformation of a vector space V onto a vector space Z. Define the mapping $\bar{T} : V/\mathsf{N}(T) \to Z$ by $\bar{T}(v + \mathsf{N}(T)) = T(v)$ for any coset $v + \mathsf{N}(T)$ in $V/\mathsf{N}(T)$.

(a) Prove that $\bar{T}$ is well-defined; that is, prove that if $v + \mathsf{N}(T) = v' + \mathsf{N}(T)$, then $T(v) = T(v')$

(b) Prove that $\bar{T}$ is linear

(c) Prove that $\bar{T}$ is an isomorphism

(d) Prove that the diagram shown below commutes; that is, prove that $T = \bar{T}\eta$



Solution: Denote $N = \mathsf{N}(T)$.

(a) Let $v, v' \in V$ be such that $v + N = v' + N$. Then by Question 1.3.31(b) in HW1, $v - v' \in N = \mathsf{N}(T)$, so $T(v - v') = 0$, $T(v) = T(v')$. Hence $\bar{T}$ is well-defined.

(b) Let $v + N, v' + N \in V/N$ and $\alpha \in \mathbb{F}$. Then $\bar{T}(\alpha(v+N) + (v'+N)) = \bar{T}((\alpha v + v') + N) = T(\alpha v + v') = \alpha T(v) + T(v') = \alpha\bar{T}(v+N) + \bar{T}(v'+N)$. As $v+N, v'+N, \alpha$ are arbitrary, $\bar{T}$ is linear.

(c) Let $v + N \in V/N$ be such that $\bar{T}(v+N) = 0$. Then $T(v) = \bar{T}(v+N) = 0$, so $v \in \mathsf{N}(T) = N$, $v + N = 0 + N$ is the zero coset. As $v + N$ is arbitrary, $\mathsf{N}\left(\bar{T}\right) = \{0 + N\}$.

   Let $z \in Z$. Since T is onto, there exists $v \in V$ such that $z = T(v) = \bar{T}(v+N) \in \mathsf{R}\left(\bar{T}\right)$. As z is arbitrary, $\mathsf{R}\left(\bar{T}\right) = Z$.

   So $\bar{T}$ is a linear transformation that is one-to-one and onto, and so an isomorphism.

(d) As defined in Question 2.1.40, we have that $\eta : V \to V/N$ is the canonical projection map to the quotient space $\eta(v) = v + W$ for $v \in V$. Then for each $v \in V$, we have $T(v) = \bar{T}(v+N) = \bar{T}(\eta v) = (\bar{T}\eta)(v)$. As v is arbitrary, $T = \bar{T}\eta$.

Note

You can replace Z with $\mathsf{R}(T) \subseteq Z$ in case T is not onto.

Compare this result with the dimension theorem and the results in Question 2.1.40. See also the first isomorphism theorem (e.g. for group) and the splitting lemma.

2.4.25. Let V be a nonzero vector space over a field $\mathbb{F}$, and suppose that S is a basis for V. Let $\mathcal{C}(S, \mathbb{F})$ denote the vector space of all function $f \in \mathcal{F}(S, \mathbb{F})$ such that $f(s) = 0$ for all but a finite number of vectors in S. Let $\Psi : \mathcal{C}(S, \mathbb{F}) \to V$ be defined by $\Psi(f) = 0$ if f is the zero function, and

$$\Psi(f) = \sum_{s \in S,\, f(s) \neq 0} f(s)s$$

otherwise. Prove that Ψ is an isomorphism. Thus every nonzero vector space can be viewed as a space of functions.

Solution: We first show that Ψ is linear. Let $f, g \in \mathcal{C}(S, \mathbb{F})$ and $\alpha \in \mathbb{F}$. Then there exist finite subsets $S_f, S_g \subseteq S$ such that $f = 0$ on $S \setminus S_f$ and $g = 0$ on $S \setminus S_g$. As S_f, S_g are finite, so is $S_f \cup S_g$. By definition, we have $\alpha S_f + S_g = 0$ on $S \setminus (S_f \cup S_g)$. So $\Psi(\alpha f + g) = \sum_{\substack{s \in S \\ (\alpha f+g)(s) \neq 0}} (\alpha f(s) + g(s)) \cdot s = \sum_{s \in S \setminus (S_f \cup S_g)} (\alpha f(s) + g(s)) \cdot s = \alpha \sum_{s \in S \setminus (S_f \cup S_g)} f(s) \cdot s + \sum_{s \in S \setminus (S_f \cup S_g)} g(s) \cdot s = \alpha \sum_{s \in S \setminus S_f} f(s) \cdot s + \sum_{s \in S \setminus S_g} g(s) \cdot s = \alpha \sum_{s \in S,\, f(s) \neq 0} f(s) \cdot s + \sum_{s \in S,\, g(s) \neq 0} g(s) \cdot s = \alpha \Psi(f) + \Psi(g)$. As α, f, g are arbitrary, Ψ is linear.

Let $f \in \mathcal{C}(S, \mathbb{F}) \setminus \{0\}$ be such that $\Psi(f) = 0$. Then for some $s_0 \in S$, $f(s_0) \neq 0$. Also $\Psi(f) = \sum_{s \in S,\, f(s) \neq 0} f(s) \cdot s = f(s_0) \cdot s_0 + \sum_{s \in S \setminus \{s_0\},\, f(s) \neq 0} f(s) \cdot s = 0$. Since S is a basis for V, we must have $f(s_0) = 0$. This contradicts the assumption that $f \neq 0$, so no such function f exists. Hence $\mathsf{N}(\Psi) = \{0\}$.

Let $v \in V$. Then for some $n \in \mathbb{N}$, $s_1, \ldots, s_n \in S$ distinct, and $a_1, \ldots, a_n \in \mathbb{F}$, we have $v = \sum_{i=1}^{n} a_i s_i$. Define $f : S \to \mathbb{F}$ by

$$f(s) = \begin{cases} a_i & \text{if } s = s_i \text{ for some } i \in \{1, \ldots, n\} \\ 0 & \text{otherwise} \end{cases}$$

for $s \in S$. Then for $s \in S$, $f(s) \neq 0$ only if $s \in \{s_1, \ldots, s_n\}$. So $\Psi(f) = \sum_{s \in S,\, f(s) \neq 0} f(s) \cdot s = \sum_{i=1}^{n} f(s_i) \cdot s_i = v$. This implies that $v \in \mathsf{R}(\Psi)$. As v is arbitrary, $\mathsf{R}(\Psi) = V$.

So Ψ is a linear transformation that is one-to-one and onto, and so an isomorphism.

2.5.1. Label the following statements as true or false.

(a) Suppose that $\beta = \{x_1, \ldots, x_n\}$ and $\beta' = \{x'_1, \ldots, x'_n\}$ are ordered bases for a vector space and Q is the change of coordinate matrix that changes β'-coordinates into β-coordinates. Then the jth column of Q is $[x_j]_{\beta'}$

(b) Every change of coordinate matrix is invertible

(c) Let T be a linear operator on a finite-dimensional vector space V, let β and β' be ordered bases for V, and let Q be the change of coordinate matrix that changes β'-coordinates into β-coordinates. Then $[T]_\beta = Q[T]_{\beta'}Q^{-1}$

(d) The matrices $A, B \in M_{n \times n}(\mathbb{F})$ are called similar if $B = Q^{\mathsf{T}}AQ$ for some $Q \in M_{n \times n}(\mathbb{F})$

(e) Let T be a linear operator on a finite-dimensional vector space V. Then for any ordered bases β and γ for V, $[T]_\beta$ is similar to $[T]_\gamma$

Solution:

a) False unless $\beta = \beta'$ b) True c) True d) False e) True

2.5.11. Let V be a finite-dimensional vector space with ordered bases α, β and γ.

(a) Prove that if Q and R are the change of coordinate matrices that change α-coordinates into β-coordinates and β-coordinates into γ-coordinates, respectively, then RQ is the change of coordinate matrix that changes α-coordinates into γ-coordinates.

(b) Prove that if Q changes α-coordinates into β-coordinates, then Q^{-1} changes β-coordinates into α-coordinates.

Solution:

(a) By definition, $Q = [\mathrm{Id}]_\alpha^\beta$ and $R = [\mathrm{Id}]_\beta^\gamma$. Hence $RQ = [\mathrm{Id}]_\beta^\gamma [\mathrm{Id}]_\alpha^\beta = [\mathrm{Id}]_\alpha^\gamma$ is the change of coordinate matrix from α-coordinates to γ-coordinates.

(b) Since $Q^{-1}Q = I = [\mathrm{Id}]_\alpha^\alpha = [\mathrm{Id}]_\beta^\alpha [\mathrm{Id}]_\alpha^\beta = [\mathrm{Id}]_\beta^\alpha Q$, by the uniqueness of matrix inverse we must have $Q^{-1} = [\mathrm{Id}]_\beta^\alpha$ being the change of coordinate matrix from β-coordinates to α-coordinates.

2.5.13. Let V be a finite-dimensional vector space over a field $\mathbb{F}$, and let $\beta = \{x_1, \ldots, x_n\}$ be an ordered basis for V. Let Q be an $n \times n$ invertible matrix with entries from $\mathbb{F}$. Define

$$x'_j = \sum_{i=1}^{n} Q_{ij} x_j$$

for $i \leq j \leq n$, and set $\beta' = \{x'_1, \ldots, x'_n\}$. Prove that β' is a basis for V and hence Q is the change of coordinate matrix changing β'-coordinates into β-coordinates.

Solution: Let $a_1, \ldots, a_n \in \mathbb{F}$ be such that $\sum_{j=1}^n a_j x'_j = 0$. Then $\sum_{j=1}^n \sum_{i=1}^n Q_{ij} a_j x_i = \sum_{i=1}^n (Qa)_i x_i = 0$ with $a = \begin{pmatrix} a_1 & \ldots & a_n \end{pmatrix}^\mathsf{T} \in \mathbb{F}^n$. As β is a basis, we must have $(Qa)_i = 0$ for all $i \in \{1, \ldots, n\}$ and so $Qa = 0$ is the zero vector of $\mathbb{F}^n$. Since $Q \in M_{n\times n}(\mathbb{F})$ is invertible, we have $a = Q^{-1}(Qa) = 0$, so $a_1 = \ldots = a_n = 0$. This implies that $\{x'_1, \ldots, x'_n\}$ is linearly independent. In particular, $x'_1, \ldots, x'_n$ are distinct.

Since β is a basis of V, we have $\dim(V) = |\beta| = n = |\{x'_1, \ldots, x'_n\}| = |\beta'|$. As β is linearly independent, β' is also a basis of V.

By definition of β', $Q = [\mathrm{Id}]_{\beta'}^{\beta}$ is the change of coordinate matrix from β'-coordinates to β-coordinates.