MATH2040A Homework 4
Reference Solutions

1 Compulsory Part

2.2.3. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by $T(a_1, a_2) = (a_1 - a_2, a_1, 2a_1 + a_2)$. Let β be the standard ordered basis for $\mathbb{R}^2$ and $\gamma = \{ (1,1,0), (0,1,1), (2,2,3) \}$. Compute $[T]_\beta^\gamma$. If $\alpha = \{ (1,2), (2,3) \}$, compute $[T]_\alpha^\gamma$.

Solution: By definition, $\beta = \{ (1,0), (0,1) \}$.

By direct computation,

$$T(1,0) = (1,1,2) = -\frac{1}{3} \cdot (1,1,0) + 0 \cdot (0,1,1) + \frac{2}{3} \cdot (2,2,3)$$
$$T(0,1) = (-1,0,1) = -1 \cdot (1,1,0) + 1 \cdot (0,1,1) + 0 \cdot (2,2,3)$$

So $[T]_\beta^\gamma = \begin{pmatrix} -\frac{1}{3} & -1 \\ 0 & 1 \\ \frac{2}{3} & 0 \end{pmatrix}$.

For $\alpha = \{ (1,2), (2,3) \}$,

$$T(1,2) = (-1,1,4) = -\frac{7}{3} \cdot (1,1,0) + 2 \cdot (0,1,1) + \frac{2}{3} \cdot (2,2,3)$$
$$T(2,3) = (-1,2,7) = -\frac{11}{3} \cdot (1,1,0) + 3 \cdot (0,1,1) + \frac{4}{3} \cdot (2,2,3)$$

So $[T]_\alpha^\gamma = \begin{pmatrix} -\frac{7}{3} & -\frac{11}{3} \\ 2 & 3 \\ \frac{2}{3} & \frac{4}{3} \end{pmatrix}$.

2.2.5. Let

$$\alpha = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$
$$\beta = \{ 1, x, x^2 \}$$
$$\gamma = \{1\}$$

(a) Define $T : M_{2\times 2}(\mathbb{F}) \to M_{2\times 2}(\mathbb{F})$ by $T(A) = A^\mathsf{T}$. Compute $[T]_\alpha$

(b) Define $T : \mathsf{P}_2(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ by $T(f(x)) = \begin{pmatrix} f'(0) & 2f(1) \\ 0 & f''(3) \end{pmatrix}$ where $'$ denote differential. Compute $[T]_\beta^\alpha$

(c) Define $T : M_{2\times 2}(\mathbb{F}) \to \mathbb{F}$ by $T(A) = \operatorname{tr} A$. Compute $[T]_\alpha^\gamma$

(d) Define $T : \mathsf{P}_2(\mathbb{R}) \to \mathbb{R}$ by $T(f(x)) = f(2)$. Compute $[T]_\beta^\gamma$

(e) If $A = \begin{pmatrix} 1 & -2 \\ 0 & 4 \end{pmatrix}$, compute $[A]_\alpha$

(f) If $f(x) = 3 - 6x + x^2$, compute $[f(x)]_\beta$

(g) For $a \in \mathbb{F}$, compute $[a]_\gamma$

Solution:

(a) As $T\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}$, $T\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix} = \begin{pmatrix}0 & 0\\1 & 0\end{pmatrix}$, $T\begin{pmatrix}0 & 0\\1 & 0\end{pmatrix} = \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}$, $T\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix} = \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}$, we have $[T]_\alpha = \begin{pmatrix}1 & 0 & 0 & 0\\0 & 0 & 1 & 0\\0 & 1 & 0 & 0\\0 & 0 & 0 & 1\end{pmatrix}$

(b) As $T(1) = \begin{pmatrix}0 & 2\\0 & 0\end{pmatrix} = 2\cdot\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}$, $T(x) = \begin{pmatrix}1 & 2\\0 & 0\end{pmatrix} = 1\cdot\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} + 2\cdot\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}$, $T(x^2) = \begin{pmatrix}0 & 2\\0 & 2\end{pmatrix} = 2\cdot\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix} + 2\cdot\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}$,

we have $[T]_\beta^\alpha = \begin{pmatrix}0 & 1 & 0\\2 & 2 & 2\\0 & 0 & 0\\0 & 0 & 2\end{pmatrix}$

(c) As $T\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} = 1$, $T\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix} = 0$, $T\begin{pmatrix}0 & 0\\1 & 0\end{pmatrix} = 0$, $T\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix} = 1$, we have $[T]_\alpha^\gamma = \begin{pmatrix}1 & 0 & 0 & 1\end{pmatrix}$

(d) As $T(1) = 1$, $T(x) = 2 = 2\cdot 1$, $T(x^2) = 4 = 4\cdot 1$, we have $[T]_\beta^\gamma = \begin{pmatrix}1 & 2 & 4\end{pmatrix}$

(e) As $A = 1\cdot\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} - 2\cdot\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix} + 0\cdot\begin{pmatrix}0 & 0\\1 & 0\end{pmatrix} + 4\cdot\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}$, we have $[A]_\alpha = \begin{pmatrix}1\\-2\\0\\4\end{pmatrix}$

(f) As $f(x) = 3\cdot 1 - 6\cdot x + 1\cdot x^2$, we have $[f(x)]_\beta = \begin{pmatrix}3\\-6\\1\end{pmatrix}$

(g) As $a = a\cdot 1$, we have $[a]_\gamma = \begin{pmatrix}a\end{pmatrix}$

2.2.10. Let V be a vector space with the ordered basis $\beta = \{ v_1, \dots, v_n \}$. Define $v_0 = 0$. Let $T : V \to V$ be a linear transformation such that $T(v_j) = v_j + v_{j-1}$ for $j \in \{ 1, \dots, n \}$. Compute $[T]_\beta$

Solution: By definition, we have

- $T(v_1) = v_1 + v_0 = v_1 = 1\cdot v_1$
- $T(v_k) = v_k + v_{k-1} = 1\cdot v_{k-1} + 1\cdot v_k$ for $k \in \{ 2, \dots, n \}$

So we have $[T]_\beta = \begin{pmatrix}1 & 1 & 0 & \dots & 0\\0 & 1 & 1 & \dots & 0\\0 & 0 & \ddots & 1 & 0\\0 & 0 & 0 & 1 & 1\\0 & 0 & 0 & 0 & 1\end{pmatrix}$

2.2.13. Let V and W be vector spaces, and let T and U be nonzero linear transformations from V into W. If $\mathsf{R}(T) \cap \mathsf{R}(U) = \{0\}$, prove that $\{T, U\}$ is a linearly independent subset of $\mathcal{L}(V, W)$.

Solution: Let $\alpha, \beta \in \mathbb{F}$ be such that $\alpha T + \beta U = 0$. Then for all $v \in V$, $(\alpha T + \beta U)(v) = \alpha T(v) + \beta U(v) = 0$, $T(\alpha v) = \alpha T(v) = -\beta U(v) = U(-\beta v)$, so $T(\alpha v) = U(-\beta v) \in \mathsf{R}(T) \cap \mathsf{R}(U) = \{0\}$, hence $\alpha T(v) = T(\alpha v) = U(-\beta v) = -\beta U(v) = 0$. As v is arbitrary, this implies that $\alpha T = -\beta U = 0$. As T, U are nonzero, we have $\alpha = \beta = 0$. This implies that $\{ T, U \}$ is linearly independent.

Note

You can also show that neither of T, U can be a multiple of the other one.

2.2.14. Let $V = \mathsf{P}(\mathbb{R})$, and for $j \geq 1$ define $T_j(f(x)) = f^{(j)}(x)$, where $f^{(j)}(x)$ is the jth derivative of $f(x)$. Prove that the set $\{ T_1, \dots, T_n \}$ is a linearly independent subset of $\mathcal{L}(V)$ for any positive integer n.

Solution: Let $a_1, \dots, a_n \in \mathbb{R}$ be such that $\sum_{i=1}^n a_i T_i = 0$. Then for all $f \in V = \mathsf{P}(\mathbb{R})$, $\left(\sum_{i=1}^n a_i T_i \right)(f) = \sum_{i=1}^n a_i T_i(f) = 0$. In particular, $0 = \left(\sum_{i=1}^n a_i T_i \right)(x^n) = \sum_{i=1}^n a_i T_i(x^n) = \sum_{i=1}^n a_i \frac{\mathrm{d}^i}{\mathrm{d}x^i} x^n = \sum_{i=1}^n a_i \frac{n!}{(n-i)!} x^{n-i}$ where $\frac{n!}{(n-i)!} \neq 0$. Since $x^{n-1}, \dots, x^0 = 1$ are of distinct orders, by comparing coefficients we have that $a_i \frac{n!}{(n-i)!} = 0$ for all $i \in \{ 1, \dots, n \}$. As $\frac{n!}{(n-i)!} \neq 0$, $a_i = 0$ for all $i \in \{ 1, \dots, n \}$. This means that the set $\{ T_1, \dots, T_n \}$ is linearly independent.

2.3.3. Let $g(x) = 3 + x$. Let $T : \mathsf{P}_2(\mathbb{R}) \to \mathsf{P}_2(\mathbb{R})$ and $U : \mathsf{P}_2(\mathbb{R}) \to \mathbb{R}^3$ be the linear transformations respectively defined by $T(f(x)) = f'(x)g(x) + 2f(x)$ and $U(a + bx + cx^2) = (a + b, c, a - b)$. Let β and γ be the standard orded bases of $\mathsf{P}_2(\mathbb{R})$ and $\mathbb{R}^3$, respectively.

(a) Compute $[U]_\beta^\gamma$, $[T]_\beta$, and $[UT]_\beta^\gamma$ directly. Then use Theorem 2.11 to verify your result.

(b) Let $h(x) = 3 - 2x + x^2$. Compute $[h(x)]_\beta$ and $[U(h(x))]_\gamma$. Then use $[U]_\beta^\gamma$ from (a) and Theorem 2.14 to verify your answer.

Solution:

(a) As $U(1) = (1,0,1) = 1 \cdot (1,0,0) + 1 \cdot (0,0,1)$, $U(x) = (1,0,-1) = 1 \cdot (1,0,0) - 1 \cdot (0,0,1)$, $U(x^2) = (0,1,0)$, we have $[U]_\beta^\gamma = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$.

As $T(1) = 2 = 2 \cdot 1$, $T(x) = g(x) + 2x = 3 + 3x = 3 \cdot 1 + 3 \cdot x$, $T(x^2) = 2xg(x) + x^2 = 4x^2 + 6x = 6 \cdot x + 4 \cdot x^2$, we have $[T]_\beta = \begin{pmatrix} 2 & 3 & 0 \\ 0 & 3 & 6 \\ 0 & 0 & 4 \end{pmatrix}$

As $UT(1) = U(2) = (2,0,2) = 2 \cdot (1,0,0) + 2 \cdot (0,0,1)$, $UT(x) = U(3+3x) = (6,0,0) = 6 \cdot (1,0,0)$, $UT(x^2) = U(6x + 4x^2) = (6,4,-6) = 6 \cdot (1,0,0) + 4 \cdot (0,1,0) - 6 \cdot (0,1,1)$, we have $[UT]_\beta^\gamma = \begin{pmatrix} 2 & 6 & 6 \\ 0 & 0 & 4 \\ 2 & 0 & -6 \end{pmatrix}$

By Theorem 2.11, we have $[UT]_\beta^\gamma = [U]_\beta^\gamma [T]_\beta = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 3 & 0 \\ 0 & 3 & 6 \\ 0 & 0 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 6 & 6 \\ 0 & 0 & 4 \\ 2 & 0 & -6 \end{pmatrix}$, which is consistent with our result.

(b) Since $h(x) = 3 - 2x + x^2 = 3 \cdot 1 - 2 \cdot x + 1 \cdot x^2$, we have $[h(x)]_\beta = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$.

Also, $U(h(x)) = U(3 - 2x + x^2) = (1,1,5) = 1 \cdot (1,0,0) + 1 \cdot (0,1,0) + 5 \cdot (0,0,1)$, we have $[U(h(x))]_\gamma = \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix}$.

By Theorem 2.14, $[U(h(x))]_\gamma = [U]_\beta^\gamma [h(x)]_\beta = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix}$, which is consistent with our result.

2.3.11. Let V be a vector space, and let $T : V \to V$ be linear. Prove that $T^2 = 0$ if and only if $\mathsf{R}(T) \subseteq \mathsf{N}(T)$.

Solution: Suppose $\mathsf{R}(T) \subseteq \mathsf{N}(T)$. Then for all $v \in V$, $Tv \in \mathsf{R}(T) \subseteq \mathsf{N}(T)$, so $T^2 v = T(Tv) = 0$. As v is arbitrary, $T^2 = 0$.

Suppose $T^2 = 0$. Then for all $v \in \mathsf{R}(T)$, there exists $w \in V$ such that $v = Tw$, $Tv = T^2 w = 0$, $v \in \mathsf{N}(T)$. As v is arbitrary, we have $\mathsf{R}(T) \subseteq \mathsf{N}(T)$.

Therefore $T^2 = 0$ if and only if $\mathsf{R}(T) \subseteq \mathsf{N}(T)$.

2.3.16. Let V be a finite-dimensional vector space, and let $T : V \to V$ be linear.

(a) If $\operatorname{rank}(T) = \operatorname{rank}(T^2)$, prove that $\mathsf{R}(T) \cap \mathsf{N}(T) = \{0\}$. Deduce that $V = \mathsf{R}(T) \oplus \mathsf{N}(T)$.

(b) Prove that $V = \mathsf{R}(T^k) \oplus \mathsf{N}(T^k)$ for some positive integer k.

Solution:

(a) As $\text{rank}(T) = \text{rank}(T^2)$, then by dimension theorem we have $\text{rank}(T) + \text{nullity}(T) = \dim(V) = \text{rank}(T^2) + \text{nullity}(T^2)$ and so $\text{nullity}(T) = \text{nullity}(T^2)$ as all quantities are finite.

For each $v \in \mathsf{N}(T)$, we have $Tv = 0$ and so $T^2v = T(Tv) = 0$ and so $v \in \mathsf{N}(T^2)$. This means that $\mathsf{N}(T) \subseteq \mathsf{N}(T^2)$. As these two subspaces have the same dimension, $\mathsf{N}(T) = \mathsf{N}(T^2)$.

Let $v \in \mathsf{R}(T) \cap \mathsf{N}(T)$. Then there exists $w \in V$ such that $Tw = v$ and $Tv = 0$, so $T^2(w) = T(Tv) = 0$, $w \in \mathsf{N}(T^2) = \mathsf{N}(T)$. This implies that $v = Tw = 0$. As v is arbitrary, $\mathsf{R}(T) \cap \mathsf{N}(T) \subseteq \{0\}$. It is easy to see that $\{0\} \subseteq \mathsf{R}(T) \cap \mathsf{N}(T)$, so we have $\mathsf{R}(T) \cap \mathsf{N}(T) = \{0\}$.

The fact that $V = \mathsf{R}(T) \oplus \mathsf{N}(T)$ then comes from Question 2.1.35(b) in HW3.

(b) By the previous part, it suffices to show that $\mathsf{R}(T^k) = \mathsf{R}((T^k)^2) = \mathsf{R}(T^{2k})$ for some $k \in \mathbb{Z}^+$.

Let $v \in \mathsf{R}(T^{k+1})$ for some $k \in \mathbb{Z}^+$. Then there exists $w \in V$ such that $v = T^{k+1}(w) = T^k(T(w)) \in \mathsf{R}(T^k)$. As v, k are arbitrary, $\mathsf{R}(T^{k+1}) \subseteq \mathsf{R}(T^k)$ for all $k \in \mathbb{Z}^+$. In particular, we have the decreasing chain $\mathsf{R}(T) \supseteq \mathsf{R}(T^2) \supseteq \ldots$ and so $\text{rank}\, T \geq \text{rank}\, T^2 \geq \ldots$.

Since V is finite-dimensional, $\text{rank}\, T \leq \dim(V) < \infty$. Then the set $\{0, \ldots, \dim(V)\}$ is a finite set, so by pigeonhole principle there exists $\text{rank}\, T^k = \text{rank}\, T^j$ for some $k, j \in \mathbb{Z}^+$, $k < j$. By the monotonicity of $\{\text{rank}\, T^n\}$, we have that $\text{rank}\, T^k = \text{rank}\, T^{k+1}$. By the decreasing chain on the range, we have that $\mathsf{R}(T^k) = \mathsf{R}(T^{k+1})$.

We now show by induction (on n) that $\mathsf{R}(T^k) = \mathsf{R}(T^n)$ for all $n \geq k$. The base cases come from the definition of k. Suppose $\mathsf{R}(T^k) = \mathsf{R}(T^l)$ for some $l \geq k+1$. Let $v \in \mathsf{R}(T^k) = \mathsf{R}(T^{k+1})$. Then for some $w \in V$, $v = T^{k+1}w = T(T^kw)$. As $T^kw \in \mathsf{R}(T^k) = \mathsf{R}(T^l)$, there exists $x \in V$ such that $T^kw = T^lx$, so $v = T(T^kw) = T(T^lx) = T^{l+1}x \in \mathsf{R}(T^{l+1})$. As v is arbitrary, $\mathsf{R}(T^k) \subseteq \mathsf{R}(T^{l+1})$. By the decreasing chain, we have $\mathsf{R}(T^k) \supseteq \mathsf{R}(T^{l+1})$ and so $\mathsf{R}(T^k) = \mathsf{R}(T^{l+1})$.

By induction, we have $\mathsf{R}(T^k) = \mathsf{R}(T^n)$ for all $n \geq k$. As $k \geq 1$, we have $2k \geq k$ and so $\mathsf{R}(T^k) = \mathsf{R}(T^{2k})$. Therefore $V = \mathsf{R}(T^k) \oplus \mathsf{N}(T^k)$.

Note

For part (b), you can also use monotone convergence theorem on $\{\text{rank}\, T^n\}_{n \in \mathbb{Z}^+}$ (which gives stability of the chain by definition), or work on nullity instead of rank. This proof also gives you an estimate on the minimal k that has this property: $k \leq \dim(V)$.

2 Optional Part

2.2.1. Label the following statements as true or false. Assume that V and W are finite-dimensional vector spaces with ordered bases β and γ, respectively, and $T, U : V \to W$ are linear transformations.

(a) For any scalar a, $aT + U$ is a linear transformation from V to W.

(b) $[T]_\beta^\gamma = [U]_\beta^\gamma$ implies that $T = U$.

(c) If $m = \dim(V)$ and $n = \dim(W)$, then $[T]_\beta^\gamma$ is an $m \times n$ matrix.

(d) $[T + U]_\beta^\gamma = [T]_\beta^\gamma + [U]_\beta^\gamma$.

(e) $\mathcal{L}(V, W)$ is a vector space.

(f) $\mathcal{L}(V, W) = \mathcal{L}(W, V)$.

Solution:

a) True
b) True
c) False. It should be $n \times m$
d) True
e) True
f) False unless $V = W$

2.2.2. Let β and γ be the standard ordered bases for $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. For each linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$, compute $[T]_\beta^\gamma$.

(a) $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(a_1, a_2) = (2a_1 - a_2, 3a_1 + 4a_2, a_1)$.

(b) $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(a_1, a_2, a_3) = (2a_1 + 3a_2 - a_3, a_1 + a_3)$

(c) $T:\mathbb{R}^3 \to \mathbb{R}$ defined by $T(a_1,a_2,a_3) = 2a_1 + a_2 - 3a_3$.
(d) $T:\mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(2a_2 + a_3, -a_1 + 4a_2 + 5a_3, a_1 + a_3)$
(e) $T:\mathbb{R}^n \to \mathbb{R}^n$ defined by $T(a_1,\dots,a_n) = (a_1,\dots,a_1)$
(f) $T:\mathbb{R}^n \to \mathbb{R}^n$ defined by $T(a_1,\dots,a_n) = (a_n,\dots,a_1)$
(g) $T:\mathbb{R}^n \to \mathbb{R}$ defined by $T(a_1,\dots,a_n) = a_1 + a_n$

Solution:

a) $\begin{pmatrix} 2 & -1 \\ 3 & 4 \\ 1 & 0 \end{pmatrix}$ b) $\begin{pmatrix} 2 & 3 & -1 \\ 1 & 0 & 1 \end{pmatrix}$ c) $\begin{pmatrix} 2 & 1 & -3 \end{pmatrix}$ d) $\begin{pmatrix} 0 & 2 & 1 \\ -1 & 4 & 5 \\ 1 & 0 & 1 \end{pmatrix}$

e) $\begin{pmatrix} 1 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \dots & 0 \end{pmatrix}$ f) $\begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & \dots & 0 & 1 & 0 \\ \dots & 0 & 1 & 0 & \vdots \\ 0 & 1 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix}$ g) $\begin{pmatrix} 1 & 0 & \dots & 0 & 1 \end{pmatrix}$

2.2.9. Let V be the vector space of complex numbers over the field $\mathbb{R}$. Define $T : V \to V$ by $T(z) = \overline{z}$, where $\overline{z}$ is the complex conjugate of z. Prove that T is linear, and compute $[T]_\beta$, where $\beta = \{ 1, i \}$.

Solution:

(a) Let $v, w \in V$, $\alpha \in \mathbb{R}$. Then there exist $v_1, v_2, w_1, w_2 \in \mathbb{R}$ such that $v = v_1 + iv_2$ and $w = w_1 + iw_2$, so $T(v) = v_1 - iv_2$, $T(w) = w_1 - iw_2$, and $T(\alpha v + w) = T((\alpha v_1 + w_1) + i(\alpha v_2 + w_2)) = \overline{(\alpha v_1 + w_1) + i(\alpha v_2 + w_2)} = \alpha(v_1 - iv_2) + (w_1 - iw_2) = \alpha T(v) + T(w)$. As v, w, α are arbitrary, T is linear.

(b) By direct computation, we have $T(1) = \overline{1} = 1$ and $T(i) = \overline{i} = -i = -1 \cdot i$, so $[T]_\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

2.2.11. Let V be an n-dimensional vector space, and let $T : V \to V$ be a linear transformation. Suppose that W is a T-invariant subspace of V having dimension k. Show that there is a basis β for V such that $[T]_\beta$ has the form $\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$, where A is a $k \times k$ matrix and 0 is the $(n-k) \times k$ zero matrix.

Solution: Let $\alpha \subseteq W$ be an (ordered) basis of W. Then $|\,\alpha\,| = k$. Extend α to a basis $\beta \supseteq \alpha$ of V by adding vectors at the end. Then $|\,\beta\,| = n$, $|\,\beta \setminus \alpha\,| = n - k$.

We now show that β has the desired property. Assume $\alpha = \{ \alpha_1, \dots, \alpha_k \}$ and $\beta = \{ \alpha_1, \dots, \alpha_k, \beta_{k+1}, \dots, \beta_n \}$ with $k, n \in \mathbb{N}$ (with the convention that empty range implies empty set). As W is T-invariant, for each $i \in \{ 1, \dots, k \}$ we have $T\alpha_i \in W$, which means $T\alpha_i = \sum_{j=1}^k c_j \alpha_j = \sum_{j=1}^k c_j \alpha_j + \sum_{j=k+1}^n 0 \cdot \beta_j$ and so the last $n-k$ entries of $[T\alpha_i]_\beta$ is nonzero. This implies that the bottom left $(n-k) \times k$ block of $[T]_\beta$ is all zero. Hence $[T]_\beta$ has the desired form.

2.2.16. Let V and W be vector spaces such that $\dim(V) = \dim(W)$, and let $T : V \to W$ be linear. Show that there exist ordered bases β and γ for V and W, respectively, such that $[T]_\beta^\gamma$ is a diagonal matrix.

Solution: We will suppose that $\dim(V) = \dim(W) = n < \infty$.

Let $\alpha \subseteq V$ be an (ordered) basis of $\mathsf{N}(\,T\,)$, and extend it to an (ordered) basis $\beta \supseteq \alpha$ of V. Since the sets are finite, we may assume that $\alpha = \{ v_1, \dots, v_k \}$ and $\beta = \{ v_1, \dots, v_n \}$ with $k = \operatorname{nullity} T \in \{ 0, \dots, n \}$.

Let $c_{k+1}, \dots, c_n \in \mathbb{F}$ be scalars such that $\sum_{i=k+1}^n c_i T v_i = 0$. Then $T\left(\sum_{i=k+1}^n c_i v_i\right) = 0$, $\sum_{i=k+1}^n c_i v_i \in \mathsf{N}(\,T\,)$, hence $\sum_{i=k+1}^n c_i v_i = \sum_{j=1}^k d_j v_j$, $\sum_{i=k+1}^n c_i v_i - \sum_{j=1}^k d_j v_j = 0$ for some scalars $d_1, \dots, d_k \in \mathbb{F}$. By the linear independence of the basis β, $c_i = d_j = 0$ for all i, j. This implies that $\{ Tv_i : i \in \{ k+1, \dots, n \} \}$ is a linearly independent set in W. Thus we can extend it to an (ordered) basis $\gamma = \{ w_1, \dots, w_l, Tv_{k+1}, \dots, Tv_n \}$ of W for some $l \in \mathbb{N}$ with $w_1, \dots, w_l \in W$. Since $l + (n-k) = \dim(W) = \dim(V) = n$, we must have $l = k$.

We now show that $[T]_\beta^\gamma$ is diagonal. For each $i \in \{ 1, \dots, k \}$, $v_i \in \mathsf{N}(\,T\,)$ and so $Tv_i = 0$, $[Tv_i]_\gamma = \begin{pmatrix} 0 & \dots & 0 \end{pmatrix}^\mathsf{T}$. For each $i \in \{ k+1, \dots, n \}$, Tv_i is the ith vector in γ, so $[Tv_i]_\gamma = \begin{pmatrix} 0 & \dots & 0 & 1 & 0 & \dots & 0 \end{pmatrix}^\mathsf{T}$ where only the ith entry is nonzero.

This implies that $[T]_\beta^\gamma = \begin{pmatrix} 0_{k \times k} & 0_{k \times (n-k)} \\ 0_{(n-k) \times k} & I_k \end{pmatrix}$ is diagonal.

Note

It is also possible to handle the case where $\dim(V) = \dim(W) = \infty$, but we would need to extend the concepts (e.g. diagonal matrix) beyond the scope of the course.

2.3.1. Label the following statements as true or false. In each part, V, W, and Z denote vector spaces with ordered (finite) bases α, β, and γ, respectively; $T : V \to W$ and $U : W \to Z$ denote linear transformations; and A and B denote matrices.

(a) $[UT]_\alpha^\gamma = [T]_\alpha^\beta [U]_\beta^\gamma$

(b) $[T(v)]_\beta = [T]_\alpha^\beta [v]_\alpha$ for all $v \in V$

(c) $[U(w)]_\beta = [U]_\alpha^\beta [w]_\beta$ for all $w \in W$

(d) $[\mathrm{Id}_V]_\alpha = I$

(e) $[T^2]_\alpha^\beta = ([T]_\alpha^\beta)^2$

(f) $A^2 = I$ implies that $A = I$ or $A = -I$

(g) $T = \mathsf{L}_A$ for some matrix A

(h) $A^2 = 0$ implies that $A = 0$, where 0 denotes the zero matrix

(i) $\mathsf{L}_{A+B} = \mathsf{A} + \mathsf{B}$

(j) If A is square and $A_{ij} = \delta_{ij}$ for all i and j, then $A = I$

Solution:

a) False

b) True

c) False. Note the mismatch in the bases

d) True

e) False unless $V = W$ and $\alpha = \beta$

f) False. Consider $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ for $\mathbb{R}^2$

g) True

h) False. Consider $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ for $\mathbb{R}^2$

i) True

j) True

2.3.12. Let V, W, and Z be vector spaces, and let $T : V \to W$ and $U : W \to Z$ be linear.

(a) Prove that if UT is one-to-one, then T is one-to-one. Must U also be one-to-one?

(b) Prove that if UT is onto, then U is onto. Must T also be onto?

(c) Prove that if U and T are one-to-one and onto, then UT is also.

Solution:

(a) Let $v \in\in \mathsf{N}(T)$. Then $Tv = 0$, so $UT(v) = 0$. As UT is one-to-one, $v = 0$. So T is also one-to-one.

Consider $V = W = Z$ is the real sequence space, U is the left-shift operator and T is the right-shift operator. Then $UT = \mathrm{Id}$ is the identity map and so is one-to-one, but U the left-shift operator is not one-to-one, as proven in Question 2.1.21 in HW3.

(b) Let $z \in Z$. As UT is onto, there exists $v \in V$ such that $z = UT(v) = U(T(v)) \in \mathsf{R}(U)$. As z is arbitrary, U is onto.

Consider $V = W = Z$ is the real sequence space, U is the left-shift operator and T is the right-shift operator. Then $UT = \mathrm{Id}$ is the identity map and so is onto, but T the right-shift operator is not onto, as proven in Question 2.1.21 in HW3.

(c) Suppose U, T are one-to-one and onto.

Let $z \in Z$. Then there exists $w \in W$ such that $z = Uw$. As T is onto, there exists $v \in V$ such that $Tv = w$. So $z = Uw = U(Tv) = (UT)(v) \in \mathsf{R}(UT)$. As z is arbitrary, UT is onto.

Let $v \in \mathsf{N}(UT)$. Then $0 = UT(v) = U(Tv)$. So $Tv \in \mathsf{N}(U) = \{0\}$. Hence $Tv = 0$, $v \in \mathsf{N}(T)$, $v = 0$. As v is arbitrary, UT is one-to-one.

Hence UT is one-to-one and onto.

2.3.13. Let A and B be $n \times n$ matrices. Prove that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ and $\mathrm{tr}(A) = \mathrm{tr}(A^\mathsf{T})$.

Solution:

(a) $\mathrm{tr}(AB) = \sum_{i=1}^n (AB)_{ii} = \sum_{i=1}^n \sum_{j=1}^n A_{ij}B_{ji} = \sum_{j=1}^n \sum_{i=1}^n B_{ji}A_{ij} = \sum_{j=1}^n (BA)_{jj} = \mathrm{tr}(BA)$

(b) $\mathrm{tr}(A^\mathsf{T}) = \sum_{i=1}^n (A^\mathsf{T})_{ii} = \sum_{i=1}^n A_{ii} = \mathrm{tr}(A)$

2.3.17. Let V be a vector space. Determine all linear transformations $T : V \to V$ such that $T = T^2$.

Solution:

(a) Let $T : V \to V$ be a linear map such that $T^2 = T$. Then $\mathsf{N}(T) = \mathsf{N}(T^2)$ and $\mathsf{R}(T) = \mathsf{R}(T^2)$.

Let $v \in \mathsf{R}(T) \cap \mathsf{N}(T)$. Then for some $w \in V$, $v = Tw$ and $Tv = 0$, so $v = Tw = T^2w = Tv = 0$. As v is arbitrary, $\mathsf{R}(T) \cap \mathsf{N}(T) \subseteq \{0\}$. It is trivial to see that $\{0\} \subseteq \mathsf{R}(T) \cap \mathsf{N}(T)$, so we have $\mathsf{R}(T) \cap \mathsf{N}(T) = \{0\}$.

Let $v \in V$. Then $v = Tv + (v - Tv)$ with $Tv \in \mathsf{R}(T)$ and $T(v - Tv) = Tv - T^2v = (T - T^2)v = 0$, $v - Tv \in \mathsf{N}(T)$. Hence $v \in \mathsf{R}(T) + \mathsf{N}(T)$. As v is arbitrary, $V \subseteq \mathsf{R}(T) + \mathsf{N}(T)$. It is trivial to see that $\mathsf{R}(T) + \mathsf{N}(T) \subseteq T$, so $\mathsf{R}(T) + \mathsf{N}(T) = V$.

Hence, $V = W_1 \oplus W_2$ with $W_1 = \mathsf{R}(T)$ and $W_2 = \mathsf{N}(T)$. For each $x \in W_1 = \mathsf{R}(T)$, there exists $y \in V$ such that $x = Ty = T^2y = Tx$, so $W_1 \subseteq \{ x \in V : x = Tx \}$. For each $x \in V$ and its corresponding direct sum decomposition $x = w_1 + w_2$ with $w_1 \in W_1$ and $w_2 \in W_2 = \mathsf{N}(T)$, $Tx = Tw_1 + Tw_2 = Tw_1 = w_1$, hence T is the projection on W_1 along W_2.

(b) Let $W_1, W_2 \subseteq V$ be subspaces of V such that $V = W_1 \oplus W_2$ and $T : V \to V$ be the projection on W_1 along W_2. Then by Question 2.3.16, T is linear and $W_1 = \{ x \in V : Tx = x \}$. Hence, for each $x \in V$ and its corresponding direct sum decomposition $x = w_1 + w_2$ with $w_1 \in W_1$, $w_2 \in W_2$, we have $Tx = w_1 = Tw_1 = T^2x$. So $T = T^2$.

Hence linear transformations $T : V \to V$ such that $T^2 = T$ are exactly the projection maps corresponding to some pair of subspaces that forms V as theirs direct sum.

Note

We do not use the result of Question 2.3.16 as it relies on Question 2.1.35, which relies on the finite dimension assumption on the space. However, you can notice that the proof (the part on showing the direct sum) is similar to that of Question 2.1.35. This is due to the fact that the proof of that question utilizes only the relations $\mathsf{N}(T) = \mathsf{N}(T^2)$ and $\mathsf{R}(T) = \mathsf{R}(T^2)$ (obtained from the dimension theorem and so required the finite dimension assumption), which we have here by default.

See also Question 2.1.17 in HW3.