MATH2040A Homework 1
Reference Solutions

1 Compulsory Part

1.2.8. In any vector space V, show that $(a+b)(x+y) = ax + ay + bx + by$ for any $x, y \in V$ and any $a, b \in \mathbb{F}$.

Solution: By the axioms of vector space, we have

$$(a+b)(x+y) = a(x+y) + b(x+y) = ax + ay + bx + by$$

which holds fo all $x, y \in V$ and $a, b \in \mathbb{F}$.

Note

VS8 and VS7 are used here.

1.2.13. Let V denote the set of ordered pairs of real numbers. If (a_1, a_2) and (b_1, b_2) are elements of V and $c \in \mathbb{R}$, define

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 b_2) \quad \text{and} \quad c(a_1, a_2) = (ca_1, a_2)$$

Is V a vector space over $\mathbb{R}$ with these operations? Justify your answer.

Solution: V is not a vector space as it violates VS8:

$$1 \cdot (1,3) + 1 \cdot (1,3) = (1,3) + (1,3) = (2,9) \neq (2,3) = 2 \cdot (1,3) = (1+1) \cdot (1,3)$$

Note

There are other vector space axioms that $(V, +, \cdot)$ does not satisfy, but you only need to give one such example.
It is not a valid proof if you only show that $(0,0)$ does not work as a zero vector: a vector with base set $\mathbb{R}^2$ does not necessarily have the same zero vector as $\mathbb{R}^2$. You would also have to argue that $(0,0)$ is the only appropriate choice.

1.2.17. Let $V = \{ (a_1, a_2) : a_1, a_2 \in \mathbb{F} \}$, where $\mathbb{F}$ is a field. Define addition of elements of V coordinatewise, and for $c \in \mathbb{F}$ and $(a_1, a_2) \in V$, define

$$c(a_1, a_2) = (a_1, 0)$$

Is V a vector space over $\mathbb{F}$ with these operations? Justify your answer.

Solution: V is not a vector space as it violates VS5: $1 \cdot (1,1) = (1,0) \neq (1,1)$

Note

As $\mathbb{F}$ is a field, we have $0 \neq 1$. For other cases (e.g. "$0 \neq 2$"), the characteristic of the field may need to be considered.

1.2.18. Let $V = \{ (a_1, a_2) : a_1, a_2 \in \mathbb{R} \}$. For $(a_1, a_2), (b_1, b_2) \in V$ and $c \in \mathbb{R}$ define

$$(a_1, a_2) + (b_1, b_2) = (a_1 + 2b_1, a_2 + 3b_2) \quad \text{and} \quad c(a_1, a_2) = (ca_1, ca_2)$$

Is V a vector space over $\mathbb{R}$ with these operations? Justify your answer.

Solution: V is not a vector space as it violates VS8:

$$1 \cdot (1,1) + 1 \cdot (1,1) = (1,1) + (1,1) = (3,4) \neq (2,2) = 2 \cdot (1,1) = (1+1) \cdot (1,1)$$

1.2.21. Let V and W be vector spaces over a field $\mathbb{F}$. Let

$$Z = \{\ (v, w) \ :\ v \in V \text{ and } w \in W\ \}$$

Prove that Z is a vector space over $\mathbb{F}$ with the operations

$$(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2) \quad \text{and} \quad c(v_1, w_1) = (cv_1, cw_1)$$

Solution: We verify all axioms one by one:

- (VS1) Let $x = (x_1, x_2), y = (y_1, y_2) \in V$. Then $x + y = (x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2) = (y_1 + x_1, y_2 + x_2) = (y_1, y_2) + (x_1, x_2) = y + x$.
- (VS2) Let $x = (x_1, x_2), y = (y_1, y_2), z = (z_1, z_2) \in V$. Then $(x + y) + z = ((x_1, x_2) + (y_1, y_2)) + (z_1, z_2) = (x_1 + y_1, x_2 + y_2) + (z_1, z_2) = (x_1 + y_1 + z_1, x_2 + y_2 + z_2) = (x_1, x_2) + (y_1 + z_1, y_2 + z_2) = (x_1, x_2) + ((y_1, y_2) + (z_1, z_2)) = x + (y + z)$.
- (VS3) Denote $\vec{0} = (0, 0) \in V$ with $0 \in \mathbb{F}$ being the zero element of $\mathbb{F}$. We now show that $\vec{0}$ is a zero vector of V: for all $x = (x_1, x_2) \in V$, $x + \vec{0} = (x_1, x_2) + (0, 0) = (x_1 + 0, x_2 + 0) = (x_1, x_2) = x$.
- (VS4) Let $x = (x_1, x_2) \in V$. Then $x_1, x_2 \in \mathbb{F}$. As $\mathbb{F}$ is a field, there exist elements $-x_1, -x_2 \in \mathbb{F}$ such that $x_1 + (-x_1) = x_2 + (-x_2) = 0$ with 0 being the zero element of $\mathbb{F}$. Then $y = (-x_1, -x_2) \in V$ and $x + y = (x_1, x_2) + (-x_1, -x_2) = (x_1 + (-x_1), x_2 + (-x_2)) = (0, 0) = \vec{0}$.
- (VS5) Let $x = (x_1, x_2) \in V$. Then $1 \cdot x = 1 \cdot (x_1, x_2) = (1 \cdot x_1, 1 \cdot x_2) = (x_1, x_2) = x$.
- (VS6) Let $a, b \in \mathbb{F}$ and $x = (x_1, x_2) \in V$. Then $(ab) \cdot x = (ab) \cdot (x_1, x_2) = (ab \cdot x_1, ab \cdot x_2) = a \cdot (bx_1, bx_2) = a \cdot (b \cdot (x_1, x_2)) = a \cdot (b \cdot x)$.
- (VS7) Let $a \in \mathbb{F}$ and $x = (x_1, x_2), y = (y_1, y_2) \in V$. Then $a \cdot (x + y) = a \cdot ((x_1, x_2) + (y_1, y_2)) = a \cdot (x_1 + y_1, x_2 + y_2) = (a(x_1 + y_1), a(x_2 + y_2)) = (ax_1 + ay_1, ax_2 + ay_2) = (ax_1, ax_2) + (ay_1, ay_2) = a \cdot (x_1, x_2) + a \cdot (y_1, y_2) = a \cdot x + a \cdot y$.
- (VS8) Let $a, b \in \mathbb{F}$ and $x = (x_1, x_2) \in V$. Then $(a + b) \cdot x = (a + b) \cdot (x_1, x_2) = ((a + b)x_1, (a + b)x_2) = (ax_1 + bx_1, ax_2 + bx_2) = (ax_1, ax_2) + (bx_1, bx_2) = a \cdot (x_1, x_2) + b \cdot (x_1, x_2) = a \cdot x + b \cdot x$.

As every axiom is satisfied, V is a vector space over $\mathbb{F}$ with the operations defined.

1.3.10. Prove that $W_1 = \{\ (a_1, a_2, \ldots, a_n) \in \mathbb{F}^n \ :\ a_1 + a_2 + \ldots + a_n = 0\ \}$ is a subspace of $\mathbb{F}^n$, but $W_2 = \{\ (a_1, a_2, \ldots, a_n) \in \mathbb{F}^n \ :\ a_1 + a_2 + \ldots + a_n = 1\ \}$ is not.

Solution: We will use Theorem 1.3 from textbook.

(a) As the zero vector of $\mathbb{F}^n$ is $\vec{0} = (0, 0, \ldots, 0)$ and $0 + 0 + \ldots + 0 = 0$, we have that $\vec{0} \in W_1$.

Let $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in W_1$. Then $x_1 + \ldots + x_n = y_1 + \ldots + y_n = 0$. So $(x_1 + y_1) + \ldots + (x_n + y_n) = (x_1 + \ldots + x_n) + (y_1 + \ldots + y_n) = 0 + 0 = 0$, and thus $(x_1 + y_1, \ldots, x_n + y_n) \in W_1$.

Let $x = (x_1, \ldots, x_n) \in W_1$ and $c \in \mathbb{F}$. Then $x_1 + \ldots + x_n = 0$. So $cx_1 + \ldots + cx_n = c(x_1 + \ldots + x_n) = c \cdot 0 = 0$. This implies that $c \cdot x = (cx_1, \ldots, cx_n) \in W_1$.

As $x, y \in W_1$ and $c \in \mathbb{F}$ are arbitrary, by Theorem 1.3 W_1 is a subspace of $\mathbb{F}^n$.

(b) As for the zero vector $\vec{0} = (0, \ldots, 0)$, $0 + \ldots + 0 = 0 \neq 1$, we have that $\vec{0} \notin W_2$. By Theorem 1.3, W_2 is not a subspace of $\mathbb{F}^n$.

Note

Compare W_1 with Question 1.2.13, where you cannot simply take $(0, 0) \in \mathbb{R}^2$ as the zero vector of V. This is because there is a known vector space structure (the one on $\mathbb{F}^n$) where W_1 inherits its own structure from, but in Question 1.2.13 there is no such structure ($\mathbb{R}^2$ is only used as a base set and the structure proposed for V does not depend on it).

This is also why we say "a subspace <u>of</u> a vector space".

1.3.11. Is the set $W = \{ f(x) \in \mathsf{P}(\mathbb{F}) : f(x) = 0 \text{ or } f(x) \text{ has degree } n \}$ a subspace of $\mathsf{P}(\mathbb{F})$ if $n \geq 1$? Justify your answer.

Solution: W is not a subspace of $\mathsf{P}(\mathbb{F})$ if $n \geq 1$.

By definition, $x^n + 1, x^n \in \mathsf{P}(\mathbb{F})$ are polynomials of degree n and so $x^n + 1, x_n \in W$, but $1 = (x^n + 1) - x^n$ is a polynomial of degree $0 \neq n$ and $1 \neq 0$. By the definition of subspaces (or Theorem 1.3), W is not a subspace of $\mathsf{P}(\mathbb{F})$.

1.3.17. Prove that a subset W of a vector space V is a subspace of V if and only if $W \neq \emptyset$, and, whenever $a \in \mathbb{F}$ and $x, y \in W$, then $ax \in W$ and $x + y \in W$.

Solution: Let $W \subseteq V$.

(a) Suppose W is a subspace of V. By Theorem 1.3, $0 \in W$, This implies that $W \neq \emptyset$.

Let $a \in \mathbb{F}$ and $x, y \in W$. By Theorem 1.3, $ax \in W$ and $x + y \in W$.

(b) Suppose $W \neq \emptyset$ and $ax, x + y \in W$ for all $a \in \mathbb{F}$ and $x, y \in W$. In view of Theorem 1.3, it remains to show that $0 \in W$.

As $W \neq \emptyset$, there exists some $x \in W$. Then by assumption $-x = (-1) \cdot x \in W$, and so $0 = x + (-x) \in W$.

1.3.19. Let W_1 and W_2 be subspaces of a vector space V. Prove that $W_1 \cup W_2$ is a subspace of V if and only if $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

Solution:

(a) Suppose $W_1 \cup W_2$ is a subspace of V. It suffices to show that $W_1 \not\subseteq W_2$ implies $W_2 \subseteq W_1$. So we may assume further that $W_1 \not\subseteq W_2$. This implies that there exists some $w_1 \in W_1 \setminus W_2$.

Let $w \in W_2$. Then $w, w_1 \in W_1 \cup W_2$. As $W_1 \cup W_2$ is a subspace of V, $w + w_1 \in W_1 \cup W_2$, which implies that $w + w_1 \in W_1$ or $w + w_1 \in W_2$.

If $w + w_1 \in W_2$, then $w_1 = (w + w_1) - w \in W_2$ as W_2 is a subspace of V. This contracts with the assumption, so we must have $w + w_1 \in W_1$. Then $w = (w + w_1) - w_1 \in W_1$ as W_1 is a subspace of V.

As $w \in W_2$ is arbitrary, we have $W_2 \subseteq W_1$.

(b) Suppose $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$ hold. By symmetry we may assume that it is the case that $W_1 \subseteq W_2$. Then $W_1 \cup W_2 = W_2$, which is a subspace of V.

1.3.22. Let $\mathbb{F}_1$ and $\mathbb{F}_2$ be fields. A function $g \in \mathcal{F}(\mathbb{F}_1, \mathbb{F}_2)$ is called an even function if $g(-t) = g(t)$ for each $t \in \mathbb{F}_1$ and is called an odd function $g(-t) = -g(t)$ for each $t \in \mathbb{F}_1$. Prove that the set of all even functions in $\mathcal{F}(\mathbb{F}_1, \mathbb{F}_2)$ and the set of all odd functions in $\mathcal{F}(\mathbb{F}_1, \mathbb{F}_2)$ are subspaces of $\mathcal{F}(\mathbb{F}_1, \mathbb{F}_2)$.

Solution: Let $0(t) \in \mathcal{F}(\mathbb{F}_1, \mathbb{F}_2)$ be the zero function from $\mathbb{F}_1$ to $\mathbb{F}_2$ such that $0(t) = 0$ for all $t \in \mathbb{F}_1$. Then for all $t \in \mathbb{F}_1$, $0(-t) = 0 = 0(t) = -0(t)$. This implies that $0(t)$ is both even and odd.

(a) Let $f, g \in \mathcal{F}(\mathbb{F}_1, \mathbb{F}_2)$ be even and $c \in \mathbb{F}_2$. Then for all $t \in \mathbb{F}_1$, $f(-t) = f(t)$ and $g(-t) = g(t)$.

So for all $t \in \mathbb{F}_1$, $(f + g)(-t) = f(-t) + g(-t) = f(t) + g(t) = (f + g)(t)$ and $(cf)(-t) = cf(-t) = cf(t) = (cf)(t)$, which implies that $f + g$ and cf are even functions.

By Theorem 1.3, $\{ f \in \mathcal{F}(\mathbb{F}_1, \mathbb{F}_2) \mid f \text{ is even} \}$ is a subspace of $\mathcal{F}(\mathbb{F}_1, \mathbb{F}_2)$.

(b) Let $f, g \in \mathcal{F}(\mathbb{F}_1, \mathbb{F}_2)$ be odd and $c \in \mathbb{F}_2$. Then for all $t \in \mathbb{F}_1$, $f(-t) = -f(t)$ and $g(-t) = -g(t)$.

So for all $t \in \mathbb{F}_1$, $(f + g)(-t) = f(-t) + g(-t) = -f(t) - g(t) = -(f + g)(t)$ and $(cf)(-t) = cf(-t) = -cf(t) = -(cf)(t)$, which implies that $f + g$ and cf are odd functions.

By Theorem 1.3, $\{ f \in \mathcal{F}(\mathbb{F}_1, \mathbb{F}_2) \mid f \text{ is odd} \}$ is a subspace of $\mathcal{F}(\mathbb{F}_1, \mathbb{F}_2)$.

Note

See also Question 1.3.28.

2 Optional Part

1.2.1. Label the following statements as true or false.

(a) Every vector space contains a zero vector.

(b) A vector space may have more than one zero vector.

(c) In any vector space, $ax = bx$ implies that $a = b$.

(d) In any vector space, $ax = ay$ implies that $x = y$.

(e) A vector in $\mathbb{F}^n$ may be regarded as a matrix in $M_{n\times 1}(\mathbb{F})$.

(f) An $m \times n$ matrix has m columns and n rows.

(g) In $\mathsf{P}(\mathbb{F})$, only polynomials of the same degree may be added.

(h) If f and g are polynomials of degree n, then $f + g$ is a polynomial of degree n.

(i) If f is a polynomial of degree n and c is a nonzero scalar, then cf is a polynomial of degree n.

(j) A nonzero scalar of $\mathbb{F}$ may be considered to be a polynomial in $\mathsf{P}(\mathbb{F})$ having degree zero.

(k) Two functions in $\mathcal{F}(S, \mathbb{F})$ are equal if and only if they have the same value at each element of S.

Solution:

(a) True	(b) False	(c) False	(d) False
(e) True	(f) False	(g) False	(h) False
(i) True	(j) True	(k) True	

1.2.14. Let $V = \{ (a_1, a_2, \ldots, a_n) : a_i \in \mathbb{C} \text{ for } i = 1, 2, \ldots, n \}$; so V is a vector space over $\mathbb{C}$. Is V a vector space over the field of real numbers with the operations of coordinatewise addition and multiplication?

Solution: $(V, +, \cdot)$ is a vector space over $\mathbb{R}$. This can be verified by checking every vector space axioms. We shall omit the detail proof here except pointing out the fact that $\mathbb{R}$ is a subfield of $\mathbb{C}$ and so all axioms regarding scalar multiplication with $\mathbb{C}$ still hold for $\mathbb{R}$.

1.2.15. Let $V = \{ (a_1, a_2, \ldots, a_n) : a_i \in \mathbb{R} \text{ for } i = 1, 2, \ldots, n \}$; so V is a vector space over $\mathbb{R}$. Is V a vector space over the field of complex numbers with the operations of coordinatewise addition and multiplication?

Solution: $(V, +, \cdot)$ is not a vector over $\mathbb{C}$: the scalar multiplication is not closed as $i \in \mathbb{C}$ and $(1, \ldots, 1) \in V$ but $i \cdot (1, \ldots, 1) = (i, \ldots, i) \notin V$.

1.2.20. Let V be the set of sequences $\{a_n\}$ of real numbers. For $\{a_n\}, \{b_n\} \in V$ and any real number t, define

$$\{a_n\} + \{b_n\} = \{a_n + b_n\} \quad \text{and} \quad t\{a_n\} = \{ta_n\}$$

Prove that, with these operations, V is a vector space over $\mathbb{R}$.

Solution: Similar to Question 1.2.21, We verify all axioms one by one:

- (VS1) Let $x = \{x_n\}, y = \{y_n\} \in V$. Then $x + y = \{x_n\} + \{y_n\} = \{x_n + y_n\} = \{y_n + x_n\} = \{y_n\} + \{x_n\} = y + x$.
- (VS2) Let $x = \{x_n\}, y = \{y_n\}, z = \{z_n\} \in V$. Then $(x + y) + z = (\{x_n\} + \{y_n\}) + \{z_n\} = \{x_n + y_n\} + \{z_n\} = \{x_n + y_n + z_n\} = \{x_n\} + \{y_n + z_n\} = \{x_n\} + (\{y_n\} + \{z_n\}) = x + (y + z)$.
- (VS3) Denote $\vec{0} = \{0\} \in V$. We now show that $\vec{0}$ is a zero vector of V: for all $x = \{x_n\} \in V$, $x + \vec{0} = \{x_n\} + \{0\} = \{x_n + 0\} = \{x_n\} = x$.
- (VS4) Let $x = \{x_n\} \in V$. Then for $y = \{-x_n\}$, $y \in V$ and $x + y = \{x_n\} + \{-x_n\} = \{x_n + (-x_n)\} = \{0\} = \vec{0}$.
- (VS5) Let $x = \{x_n\} \in V$. Then $1 \cdot x = 1 \cdot \{x_n\} = \{1 \cdot x_n\} = \{x_n\} = x$.
- (VS6) Let $a, b \in \mathbb{R}$ and $x = \{x_n\} \in V$. Then $(ab) \cdot x = (ab) \cdot \{x_n\} = \{abx_n\} = a \cdot \{bx_n\} = a \cdot (b \cdot \{x_n\}) = a \cdot (b \cdot x)$.

- (VS7) Let $a \in \mathbb{R}$ and $x = \{x_n\}, y = \{y_n\} \in V$. Then $a \cdot (x + y) = a \cdot (\{x_n\} + \{y_n\}) = a \cdot \{x_n + y_n\} = \{a(x_n + y_n)\} = \{ax_n + ay_n\} = \{ax_n\} + \{ay_n\} = a \cdot \{x_n\} + a \cdot \{y_n\} = a \cdot x + a \cdot y$.
- (VS8) Let $a, b \in \mathbb{R}$ and $x = \{x_n\} \in V$. Then $(a+b) \cdot x = (a+b) \cdot \{x_n\} = \{(a+b)x_n\} = \{ax_n + bx_n\} = \{ax_n\} + \{bx_n\} = a \cdot \{x_n\} + b \cdot \{y_n\} = a \cdot x + b \cdot x$.

As every axiom is satisfied, V is a vector space over $\mathbb{R}$ with the operations defined.

1.3.1. Label the following statements as true or false.

(a) If V is a vector space and W is a subset of V that is a vector space, then W is a subspace of V.

(b) The empty set is a subspace of every vector space.

(c) If V is a vector space other than the zero vector space, then V contains a subspace W such that $W \neq V$.

(d) The intersection of any two subsets of V is a subspace of V.

(e) An $n \times n$ diagonal matrix can never have more than n nonzero entries.

(f) The trace of a square matrix is the product of its diagonal entries.

(g) Let W be the xy-plane in $\mathbb{R}^3$; that is, $W = \{ (a_1, a_2, 0) : a_1, a_2 \in \mathbb{R} \}$. Then $W = \mathbb{R}^2$.

Solution:

(a) False	(b) False	(c) True
(d) False	(e) True	(f) False

(g) False. Note that W is only *isomorphic* to $\mathbb{R}^2$

1.3.8. Determine whether the following sets are subspaces of $\mathbb{R}^3$ under the operations of addition and scalar multiplication defined on $\mathbb{R}^3$. Justify your answers.

(a) $W_1 = \{ (a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 = 3a_2 \text{ and } a_3 = -a_2 \}$

(b) $W_2 = \{ (a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 = a_3 + 2 \}$

(c) $W_3 = \{ (a_1, a_2, a_3) \in \mathbb{R}^3 : 2a_1 - 7a_2 + a_3 = 0 \}$

(d) $W_4 = \{ (a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 - 4a_2 - a_3 = 0 \}$

(e) $W_5 = \{ (a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 + 2a_2 - 3a_3 = 1 \}$

(f) $W_6 = \{ (a_1, a_2, a_3) \in \mathbb{R}^3 : 5a_1^2 - 3a_2^2 + 6a_3^2 = 0 \}$

Solution:

(a) Since $0 = 3 \cdot 0$ and $0 = -0$, $\vec{0} = (0, 0, 0) \in W_1$.

Let $(a_1, a_2, a_3), (b_1, b_2, b_3) \in W_1$ and $c \in \mathbb{R}$, Then $a_1 = 3a_2$, $b_1 = 3b_2$ and $a_3 = -a_2$, $b_3 = -b_2$ and so $(a_1+b_1) = 3(a_2+b_2)$, $(a_3+b_3) = -(a_2+b_2)$ and $ca_1 = 3ca_2$, $ca_3 = -ca_2$. This implies that $(a_1, a_2, a_3) + (b_1, b_2, b_3) = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$ and $c(a_1, a_2, a_3) = (ca_1, ca_2, ca_3) \in W_1$.

By Theorem 1.3, W_1 is a subspace of $\mathbb{R}^3$.

(b) W_2 is not a subspace of $\mathbb{R}^3$ as $0 \neq 2 = 0 + 2$ and so $\vec{0} = (0, 0, 0) \notin W_2$

(c) Since $2 \cdot 0 - 7 \cdot 0 + 0 = 0$, $\vec{0} = (0, 0, 0) \in W_3$.

Let $(a_1, a_2, a_3), (b_1, b_2, b_3) \in W_3$ and $c \in \mathbb{R}$, Then $2a_1 - 7a_2 + a_3 = 2b_1 - 7b_2 + b_3 = 0$ so $2 \cdot (a_1+b_1) - 7 \cdot (a_2+b_2) + (a_3+b_3) = 0$ and $2ca_1 - 7ca_2 + ca_3 = 0$. This implies that $(a_1, a_2, a_3) + (b_1, b_2, b_3) = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$ and $c(a_1, a_2, a_3) = (ca_1, ca_2, ca_3) \in W_3$.

By Theorem 1.3, W_3 is a subspace of $\mathbb{R}^3$.

(d) Since $0 - 4 \cdot 0 - 0 = 0$, $\vec{0} = (0, 0, 0) \in W_3$.

Let $(a_1, a_2, a_3), (b_1, b_2, b_3) \in W_3$ and $c \in \mathbb{R}$, Then $a_1 - 4a_2 - a_3 = b_1 - 4b_2 - b_3 = 0$ so $2(a_1+b_1) - 4(a_2+b_2) - (a_3+b_3) = 0$ and $ca_1 - 4ca_2 - ca_3 = 0$. This implies that $(a_1, a_2, a_3) + (b_1, b_2, b_3) = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$ and $c(a_1, a_2, a_3) = (ca_1, ca_2, ca_3) \in W_4$.

By Theorem 1.3, W_4 is a subspace of $\mathbb{R}^3$.

(e) W_5 is not a subspace of $\mathbb{R}^3$ as $0 + 2 \cdot 0 - 3 \cdot 0 = 0 \neq 1$ and so $\vec{0} = (0,0,0) \notin W_5$

(f) W_6 is not a subspace of $\mathbb{R}^3$. Since

$$5 \cdot \left(\frac{1}{\sqrt{5}}\right)^2 - 3 \cdot \left(\frac{1}{\sqrt{3}}\right)^2 + 6 \cdot 0^2 = 5 \cdot 0^2 - 3 \cdot \left(-\frac{1}{\sqrt{3}}\right)^2 + 6 \cdot \left(\frac{1}{\sqrt{6}}\right) = 0$$

we have

$$\left(\frac{1}{\sqrt{5}}, \frac{1}{\sqrt{3}}, 0\right), \left(0, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{6}}\right) \in W_6$$

but

$$\left(\frac{1}{\sqrt{5}}, \frac{1}{\sqrt{3}}, 0\right) + \left(0, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{6}}\right) = \left(\frac{1}{\sqrt{5}}, 0, \frac{1}{\sqrt{6}}\right) \notin W_6$$

as

$$5 \cdot \left(\frac{1}{\sqrt{5}}\right)^2 - 3 \cdot 0^2 + 6 \cdot \left(\frac{1}{\sqrt{6}}\right)^2 = 2 \neq 0$$

1.3.23. Let W_1 and W_2 be subspaces of a vector space V.

(a) Prove, that $W_1 + W_2$ is a subspace of V that contains both W_1 and W_2.

(b) Prove that any subspace of V that contains both W_1 and W_2 must also contain $W_1 + W_2$.

Solution:

(a) Since W_1, W_2 are subspaces of V, by Theorem 1.3 $0 \in W_1$ and $0 \in W_2$, so $0 = 0 + 0 \in W_1 + W_2$.

Let $x, y \in W_1 + W_2$ and $c \in \mathbb{F}$. Then there exists $x_1, y_1 \in W_1$ and $x_2, y_2 \in W_2$ such that $x = x_1 + x_2$ and $y = y_1 + y_2$. This implies that $x + y = (x_1 + y_1) + (x_2 + y_2)$ and $cx = (cx_1) + (cx_2)$ with $x_1 + y_1, cx_1 \in W_1$ and $x_2 + y_2, cy_2 \in W_2$, which implies that $x + y, cx \in W_1 + W_2$.

By Theorem 1.3, $W_1 + W_2$ is a subspace of V.

For all $w \in W_1$, we have $w = w + 0 \in W_1 + W_2$ as $0 \in W_2$. As $w \in W_1$ is arbitrary, $W_1 \subseteq W_1 + W_2$.

For all $w \in W_2$, we have $w = 0 + w \in W_1 + W_2$ as $0 \in W_1$. As $w \in W_2$ is arbitrary, $W_2 \subseteq W_1 + W_2$.

Therefore $W_1 + W_2$ is a subspace of V that contains both W_1 and W_2.

(b) Let $U \subseteq V$ be a subspace of V that contains W_1 and W_2.

Let $x \in W_1 + W_2$. Then there exist $w_1 \in W_1$, $w_2 \in W_2$ such that $x = w_1 + w_2$. As U contains W_1 and W_2, we have $w_1, w_2 \in U$. As U is a subspace, we have that $x = w_1 + w_2 \in U$.

As $x \in W_1 + W_2$ is arbitrary, $W_1 + W_2 \subseteq U$.

As U is arbitrary, every subspace of V that contains both W_1 and W_2 also contains $W_1 + W_2$.

1.3.28. Let $\mathbb{F}$ be a field. Prove that the set W_1 of all skew-symmetric $n \times n$ matrices with entries from $\mathbb{F}$ is a subspace of $M_{n\times n}(\mathbb{F})$. Now assume that $\mathbb{F}$ is not of characteristic 2, and let W_2 be the subspace of $M_{n\times n}(\mathbb{F})$ consisting of all symmetric, $n \times n$ matrices. Prove that $M_{n\times n}(\mathbb{F}) = W_1 \oplus W_2$.

Solution:

(a) Let $0_{n\times n} \in M_{n\times n}(\mathbb{F})$ be the zero matrix. Then $0_{n\times n}^\mathsf{T} = 0_{n\times n} = -0_{n\times n}$, so $0_{n\times n}$ is a skew-symmetric matrix.

Let $A, B \in W_1$ and $c \in \mathbb{F}$. Then $A^\mathsf{T} = -A$ and $B^\mathsf{T} = -B$, so $(A + B)^\mathsf{T} = A^\mathsf{T} + B^\mathsf{T} = -A - B = -(A + B)$ and $(cA)^\mathsf{T} = cA^\mathsf{T} = -cA = -(cA)$, which implies that $A + B$, cA are skew-symmetric matrices.

By Theorem 1.3, W_1 is a subspace of $M_{n\times n}(\mathbb{F})$.

(b) Let $A \in W_1 \cap W_2$. Then A is both skew-symmetric and symmetric, so $A = A^\mathsf{T} = -A$. As $\mathbb{F}$ is not of characteristic 2, $A = 0_{n\times n}$. This implies that $W_1 \cap W_2 \subseteq \{0_{n\times n}\}$.

As W_1, W_2 are subspaces of $M_{n\times n}(\mathbb{F})$, $0_{n\times n} \in W_1 \cap W_2$, $\{0_{n\times n}\} \subseteq W_1 \cap W_2$. This implies that $W_1 \cap W_2 = \{0_{n\times n}\}$.

Let $A \in M_{n\times n}(\mathbb{F})$. Let $A_1 = \frac{1}{2}(A - A^\mathsf{T}), A_2 \in \frac{1}{2}(A + A^\mathsf{T}) \in M_{n\times n}(\mathbb{F})$. Then $A_1^\mathsf{T} = \frac{1}{2}(A - A^\mathsf{T})^\mathsf{T} = \frac{1}{2}(A^\mathsf{T} - A) = -A_1$, $A_2^\mathsf{T} = \frac{1}{2}(A + A^\mathsf{T})^\mathsf{T} = \frac{1}{2}(A^\mathsf{T} + A) = A_2$. This implies that $A_1 \in W_1$ and $A_2 \in W_2$. By definition we have that $A = A_1 + A_2 \in W_1 + W_2$. As $A \in M_{n\times n}(\mathbb{F})$ is arbitrary, $M_{n\times n}(\mathbb{F}) \subseteq W_1 + W_2$.

As $W_1, W_2 \subseteq M_{n\times n}(\mathbb{F})$, $W_1 + W_2 \subseteq M_{n\times n}(\mathbb{F})$. This implies that $M_{n\times n}(\mathbb{F}) = W_1 + W_2$.

By definition, $M_{n\times n}(\mathbb{F}) = W_1 \oplus W_2$.

1.3.30. Let W_1 and W_2 be subspaces of a vector space V. Prove that V is the direct sum of W_1 and W_2 if and only if each vector in V can be uniquely written as $x_1 + x_2$, where $x_1 \in W_1$ and $x_2 \in W_2$.

Solution:

(a) Suppose $V = W_1 \oplus W_2$. Then $V = W_1 + W_2$ and $W_1 \cap W_2 = \{0\}$. Hence for all $x \in V$ there exist $x_1 \in W_1$ and $x_2 \in W_2$ such that $x = x_1 + x_2$.

Let $x \in V$ be such that there exists $x_1, y_1 \in W_1$ and $x_2, y_2 \in W_2$ such that $x = x_1 + x_2 = y_1 + y_2$. Then $x_1 - y_1 = x_2 - y_2$. As $x_1, y_1 \in W_1$ and $x_2, y_2 \in W_2$, we have that $x_1 - y_1 \in W_1$ and $x_2 - y_2 \in W_2$. So $x_1 - y_1 = x_2 - y_2 \in W_1 \cap W_2 = \{0\}$. This implies that $x_1 - y_1 = x_2 - y_2 = 0$ and so $x_1 = y_1$, $x_2 = y_2$. Hence the decomposition for an arbitrary x is unique.

Thus, each vector in V can be uniquely written as $x_1 + x_2$ where $x_1 \in W_1$ and $x_2 \in W_2$.

(b) Suppose each vector x in V can be uniquely written as $x = x_1 + x_2$ where $x_1 \in W_1$ and $x_2 \in W_2$. By definition, this means that $V = W_1 + W_2$.

Let $x \in W_1 \cap W_2 \subseteq V$. Since W_1, W_2 are subspaces, $-x \in W_1 \cap W_2$. So $0 = 0 + 0 = x + (-x)$ with $0, x \in W_1$ and $0, -x \in W_2$. By assumption, this implies that $x = 0 = -x$ and so $W_1 \cap W_2 = \{0\}$.

By definition, we have that $V = W_1 \oplus W_2$.

1.3.31. Let W be a subspace of V.

(a) Prove that $v + W$ is a subspace of V if and only if $v \in W$

(b) Prove that $v_1 + W = v_2 + W$ if and only if $v_1 - v_2 \in W$

(c) Prove that the addition and scalar multiplication defined as

$$(v_1 + W) + (v_2 + W) = (v_1 + v_2) + W$$
$$a(v + W) = av + W$$

for $v_1, v_2, v \in V$ and $a \in \mathbb{F}$ are well-defined.

(d) Prove that the set $S = \{\, v + W \;:\; v \in V \,\}$ of all cosets of W is a vector space with the operations defined.

Solution:

(a) Let $v \in V$.

i. Suppose $v + W$ is a subspace of V. Then $0 \in v + W$. So there exists $w \in W$ such that $0 = v + w$, $w = -v$, and hence $v = -w \in W$ as W is a subspace.

ii. Suppose $v \in W$.

Let $w \in W$. Then $w - v \in W$ and so $w = v + (w - v) \in v + W$. As W is arbitrary, $W \subseteq v + W$.

Let $x \in v + W$. Then there exists $w \in W$ such that $x = v + w$. As $v, w \in W$ and W is a subspace, $x = v + w \in W$. As x is arbitrary, $v + W \subseteq W$.

So $v + W = W$, which is a subspace.

So $v + W$ is a subspace of V if and only if $v \in W$.

(b) Let $v_1, v_2 \in V$.

i. Suppose $v_1 + W = v_2 + W$. Then $v_1 = v_1 + 0 \in v_1 + W = v_2 + W$, so there exists $w \in W$ such that $v_1 = v_2 + w$. This implies that $v_1 - v_2 = w \in W$.

ii. Suppose $v_1 - v_2 \in W$.

Let $x \in v_1 + W$. Then there exists $w \in W$ such that $x = v_1 + w = v_2 + (w - (v_1 - v_2))$. As $w, v_1 - v_2 \in W$, we have that $w - (v_1 - v_2) \in W$ and so $x = v_2 + w - (v_1 - v_2) \in v_2 + W$. As x is arbitrary, $v_1 + W \subseteq v_2 + W$.

Let $x \in v_2 + W$. Then there exists $w \in W$ such that $x = v_2 + w = v_1 + (w + (v_1 - v_2))$. As $w, v_1 - v_2 \in W$, we have that $w + (v_1 - v_2) \in W$ and so $x = v_1 + w + (v_1 - v_2) \in v_1 + W$. As x is arbitrary, $v_2 + W \subseteq v_1 + W$.

Hence $v_1 + W = v_2 + W$.

So $v_1 + W = v_2 + W$ if and only if $v_1 - v_2 \in W$.

(c) i. Let $v_1, v_2, v_1', v_2' \in V$ such that $v_1 + W = v_1' + W$ and $v_2 + W = v_2' + W$. By part (b) we have that $v_1 - v_1', v_2 - v_2' \in W$, so $(v_1 + v_2) - (v_1' + v_2') = (v_1 - v_1') + (v_2 - v_2') \in W$. By part (b), $(v_1 + v_2) + W = (v_1' + v_2') + W$.

ii. Let $a \in \mathbb{F}$. Let $v, v' \in V$ be such that $v + W = v' + W$. Then by part (b), $v - v' \in W$, so $(av) - (av') = a(v - v') \in W$. By part (b), $(av) + W = (av') + W$.

Hence the operations are independent of the choice of representation, and so they are well-defined.

(d) We verify all axioms one by one:

i. Let $x + W, y + W \in S$. Then $(x + W) + (y + W) = (x + y) + W = (y + x) + W = (y + W) + (x + W)$

ii. Let $x + W, y + W, z + W \in S$. Then $((x + W) + (y + W)) + (z + W) = ((x + y) + W) + (z + W) = (x + y + z) + W = (x + W) + (((y + z)) + W) = (x + W) + ((y + W) + (z + W))$

iii. Let $\vec{0} = W = 0 + W \in S$. Then for all $x + W \in S$, $(x + W) + \vec{0} = (x + W) + (0 + W) = (x + 0) + W = x + W$

iv. Let $x + W \in S$. Then with $(-x) + W \in S$ we have $(x + W) + ((-x) + W) = (x + (-x)) + W = 0 + W = \vec{0}$

v. Let $x + W \in S$. Then $1 \cdot (x + W) = (1 \cdot x) + W = x + W$

vi. Let $a, b \in \mathbb{F}$ and $x + W \in S$. Then $(ab) \cdot (x + W) = (abx) + W = a \cdot ((bx) + W) = a \cdot (b \cdot (x + W))$

vii. Let $a \in \mathbb{F}$ and $x + W, y + W \in S$. Then $a \cdot ((x + W) + (y + W)) = a \cdot ((x + y) + W) = (a \cdot (x + y)) + W = (ax + ay) + W = (ax + W) + (ay + W) = a \cdot (x + W) + a \cdot (y + W)$

viii. Let $a, b \in \mathbb{F}$ and $x + W \in S$. Then $(a + b) \cdot (x + W) = ((a + b) \cdot x) + W = (ax + bx) + W = ((ax) + W) + ((bx) + W) = a \cdot (x + W) + b \cdot (x + W)$

As every axiom is satisfied, S is a vector space over $\mathbb{F}$ with the operations defined.

Note

$S = V/W$ is the quotient space.

The construction of quotient space is important for understanding the abstract properties of vector spaces, and similar theorems (e.g. the first isomorphism theorem) for other algebraic structures (e.g. group) regarding quotient objects also hold for quotient spaces. Unfortunately, quotient space does not seem to be in the course syllabus.